1. COMPLEX NUMBERS

A short history

The history of the complex numbers is very interesting. By the 16th Century, although no-one understood exactly what a complex number was, it was found that complex numbers were a useful tool for solving problems. Later, mathematicians tried to understand the complex numbers. This led in turn to investigations of the real numbers, the rational numbers, the integers and finally the natural numbers. So historically there was a reverse development: the more complicated system was found to be useful early on, and the study of the simplest systems were left till later.

Leibniz (1702) described complex numbers as 'that wonderful creation of an ideal world, almost an amphibian between things that are and things that are not'.

Definition

Complex numbers are defined as ordered pairs $z = (x, y)$ of real numbers x and y. We shall define addition and multiplication of these numbers shortly.

We identify the pairs $(x, 0)$ with the real numbers x. This means that the real numbers can be thought of as a subset of the complex numbers.

Complex numbers of the form $(0, y)$ are said to be **pure imaginary** numbers. The numbers x and y are now called the **real** and **imaginary** parts of z respectively, and we write $x = \mathbf{Re}\, z$, $y = \mathbf{Im}\, z$.

Equality

Two complex numbers are **equal** if they have the same real and imaginary parts. Thus

$$(x_1, y_1) = (x_2, y_2) \text{ if and only if } x_1 = x_2,\ y_1 = y_2.$$

Sum and product

The **sum** $z_1 + z_2$ and **product** $z_1 . z_2$ of $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$ are defined by:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1).(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

This last definition looks pretty weird, but we shall shortly see the reason for it.

Understanding the operations

With these definitions of addition and multiplication, we have

$$(x, y) = (x, 0) + (0, 1)(y, 0).$$

The set of complex numbers of the form $(x, 0)$ act just like the real numbers **R**. Further, setting $i = (0, 1)$ gives $z = (x, y) = x + iy$, and we have

$i^2 = (0, 1).(0, 1) = (-1, 0)$, that is, $i^2 = -1$.

Addition and multiplication can now be rewritten as the more usual:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

$$(x_1 + iy_1).(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

Comments

1. Why did we not just start here? Notice that our derivation of i is firmly based on the real numbers. There is no 'magic' about $\sqrt{(-1)}$, contrary to the use of the historical word 'imaginary'.

2. We can now perform complex operations by treating the terms as real, and substituting $i^2 = -1$.

Example

$(2 + 3i) + (4 + 5i) = (2 + 4) + (3 + 5)i = 6 + 8i$,

and

$$(2 + 3i).(4 + 5i) = (2.4 + 3i.5i) + (2.5i + 3.4i)$$
$$= (8 - 15) + (10 + 12)i = -7 + 22i.$$

QUIZ 1.1A

1. The sum of $3 + i$ and $7 + 5i$ is

2. The product of $3 + i$ and $7 + 5i$ is

3. The product of i and $-i$ is

4. The product of $1 + 2i$ and $1 - 2i$ is

CHECK

QUIZ 1.1B

With these definitions of addition and multiplication, we have { **1** }. The set of complex numbers of the form { **2** } act just like the real numbers **R**. Further, setting { **3** } gives $z = (x, y) = x + iy$, and we have { **4** }, that is, $i^2 = -1$. Addition and multiplication can now be rewritten as the more usual:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$(x_1 + iy_1).(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

Match the above missing items **1, 2, 3, 4** with the selections

(a) $i = (0, 1)$, **(b)** $(x, y) = (x, 0) + (0, 1)(y, 0)$,
(c) $i^2 = (0, 1).(0, 1) = (-1, 0)$, **(d)** $(x, 0)$.

1. ☐ **2.** ☐ **3.** ☐ **4.** ☐

CHECK

Algebraic properties

The complex numbers behave in much the same way as the real numbers.
In particular they form a **field**.

Some of the laws satisfied are:

The Commutative Laws:

$$z_1 + z_2 = z_2 + z_1; \quad z_1 z_2 = z_2 z_1;$$

The Associative Laws:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3);$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3);$$

The Distributive Law:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

These are easily proved by writing the zs in the form $x + iy$ and using the corresponding properties of the real numbers.

Other field axioms

We now use the idea of inverse to define division of complex numbers in the following way:

$$z_1/z_2 = z_1 \, . \, z_2^{-1} \quad (z_2 \neq 0).$$

Try writing this out in terms of the real and imaginary parts! Later we shall give an easier way of dividing complex numbers. Clearly division by zero is not allowed, as z_2^{-1} is undefined when $z_2 = 0$.

The expressions below follow from this definition:

$$\frac{1}{z_1 z_2} = \frac{1}{z_1} \cdot \frac{1}{z_2}, \quad \frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}, \quad \frac{z_1 z_2}{z_3 z_4} = \frac{z_1}{z_3} \cdot \frac{z_2}{z_4}$$

As mentioned earlier, complex numbers (like the reals) form a field, **C**. However, it is not possible to order **C**. Thus expressions like $z > 0$, $z_1 < z_2$ are meaningless unless the complex numbers are real.

Example

$$\frac{1}{2 - 3i} \cdot \frac{1}{1 + i} = \frac{1}{5 - i} \cdot \frac{5 + i}{5 + i} = \frac{5 + i}{26} = \frac{5}{26} + \frac{i}{26}$$

QUIZ 1.2

1. The integers form a field under addition and multiplication.

(a) True ☐ ; (b) False ☑ .

2. $3 + 2i \;>\; 2 + i$

(a) True ☑ ; (b) False ☐ ;

(c) Neither ☐ .

3. $z_1(z_2 z_3) = (z_1 z_2)z_3$ is the

________________ law.

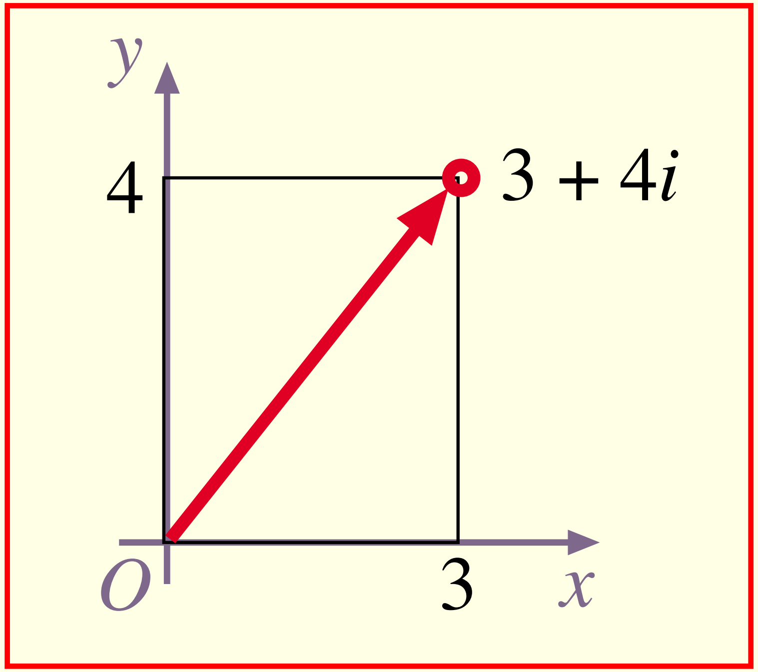
4. ${}^{2}/_{(1+i)}$ is another expression for $1 - i$.

(a) True ☑ ; (b) False ☐ .

CHECK

Cartesian coordinates

We set up the natural correspondence $z = x + iy \leftrightarrow (x, y)$ between the complex number z and the point (x, y) in the cartesian plane. Each complex number corresponds to exactly one point in the plane, and conversely. For example, $1 + 2i$ is represented by the point $(1, 2)$. Notice how this corresponds to our original 'ordered pair' definition of a complex number. We can think of the complex number z either as the point (x, y) or as the vector from the origin to this point.



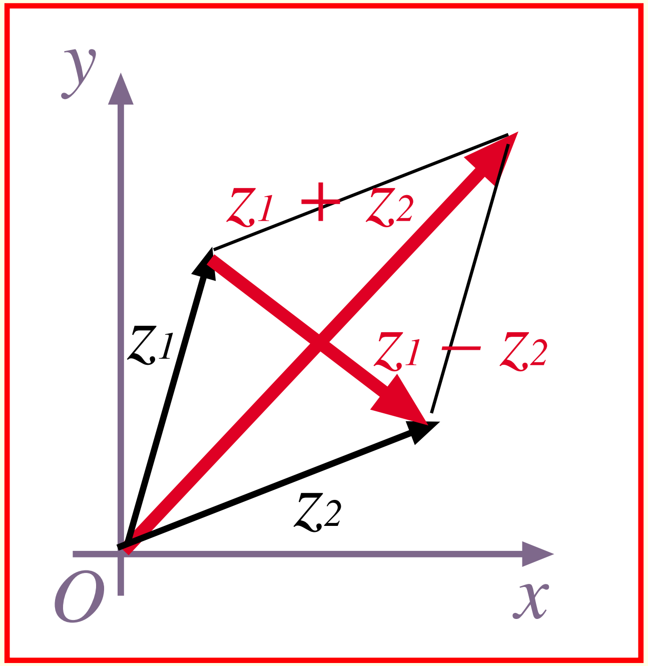
The plane of such representative points is called the **Argand diagram**, or the **complex plane** or the ***z* - plane**.

The x-axis is called the **real** axis and the y-axis is called the **imaginary** axis.

Sum and difference

We now have an immediate geometric interpretation of the sum and difference of two complex numbers (below).

Notice the direction of the arrow representing $z_1 - z_2$.



Modulus and distance

The **modulus** or **absolute value** of a complex number $z = x + iy$ is defined to be $|z| = \sqrt{x^2 + y^2}$. When $y = 0$, z is real, and we have the usual absolute value of a real number.

The (real) number $|z|$ denotes the length of the vector representing z, or the distance of the point (x, y) from the origin O. Even though $z_1 < z_2$ is meaningless, note that $|z_1| < |z_2|$ is a valid statement, meaning z_1 is closer to O than z_2.

We can extend this idea to see that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

is the distance between the points representing z_1 and z_2.

Example Find the locus of points z satisfying $|z - i| = 3$.
We are looking at points z which are distant 3 from the fixed point i. That is, a circle of radius 3 and centre $i = (0,1)$.
If we express $z = x + iy$, then $|z - i| = 3$ becomes the equation of this circle:

$$\sqrt{x^2 + (y-1)^2} = 3, \quad \text{or} \quad x^2 + (y-1)^2 = 9.$$

Modulus and conjugate

From the definition of $|z|$, we have

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2,$$

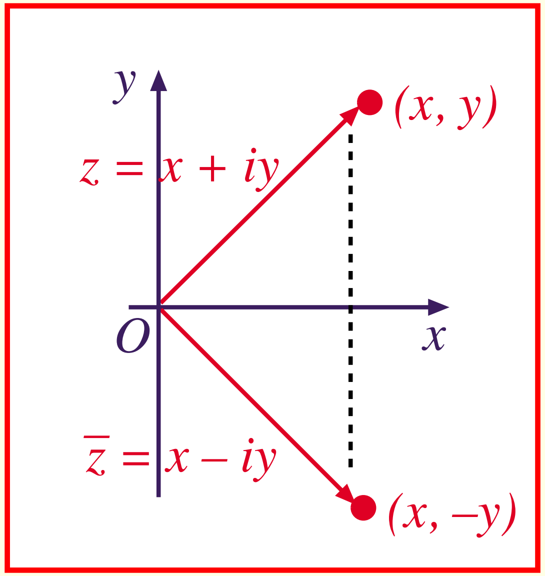
from which it follows that

$$|z| \geq |\operatorname{Re} z| \geq \operatorname{Re} z\text{, and}$$

$$|z| \geq |\operatorname{Im} z| \geq \operatorname{Im} z.$$

Argand

Jean Robert Argand (1768 – 1822) was a Swiss bookkeeper. In 1806 he published a paper associating the complex numbers with points in the complex plane. The plane of complex numbers is now often called the **Argand plane**.



The **complex conjugate** or **conjugate** of $z = x + iy$ is the number $\bar{z} = x - iy$.

The number $\bar{z}$ is represented in the complex plane by the point $(x, -y)$ – the reflection in the x-axis of the point representing z.

We observe that for all z,

$$\bar{\bar{z}} = z, \text{ and } |\bar{z}| = |z|.$$

More properties of the conjugate

The following properties are easy to establish from the definitions:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\,, \quad \overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}\,, \quad \overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}\,.$$

We also have
$$\operatorname{Re} z = \tfrac{1}{2}(z + \overline{z}),$$
$$\operatorname{Im} z = \tfrac{1}{2}(z - \overline{z}).$$

Finally, $$z\,.\,\overline{z} = |z|^2 = x^2 + y^2,$$

so for $z_2 \neq 0$, $$\frac{z_1}{z_2} = \frac{z_1 . \overline{z}_2}{z_2 . \overline{z}_2} = \frac{z_1 . \overline{z}_2}{|z_2|^2}$$

This gives us a nice way to find the quotient of two complex numbers.

Wessel

Casper Wessel (1745 – 1818) was a Norwegian surveyor. He published a paper in 1797 linking complex numbers to points in the plane, but the paper remained unnoticed for 100 years!

Example

$$\frac{1 + 8i}{2 + i} = \frac{(1 + 8i)(2 - i)}{(2 + i)(2 - i)} = \frac{10 + 15i}{5} = 2 + 3i$$

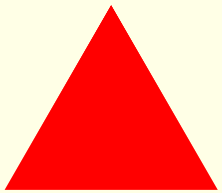
The triangle inequality

The relation $z.\bar{z} = |z|^2$ quickly gives

$|z_1.z_2| = |z_1|.|z_2|, \quad |z_1/z_2| = |z_1|/|z_2|.$

$|z_1.z_2| = |z_1|.|z_2|, \quad |z_1/z_2| = |z_1|/|z_2|.$

It is also not too hard to derive the



Triangle Inequality:

$$|\mathbf{z_1} + \mathbf{z_2}| \le |\mathbf{z_1}| + |\mathbf{z_2}|.$$

(Proof at right)

It is easy to extend this to larger finite sums:

$$\begin{aligned} |z_1 + z_2 + z_3| &\le |z_1 + z_2| + |z_3| \\ &\le |z_1| + |z_2| + |z_3|. \end{aligned}$$

Proof of the inequality

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &\le |z_1|^2 + 2|z_1|.|z_2| + |z_1|^2 \end{aligned}$$

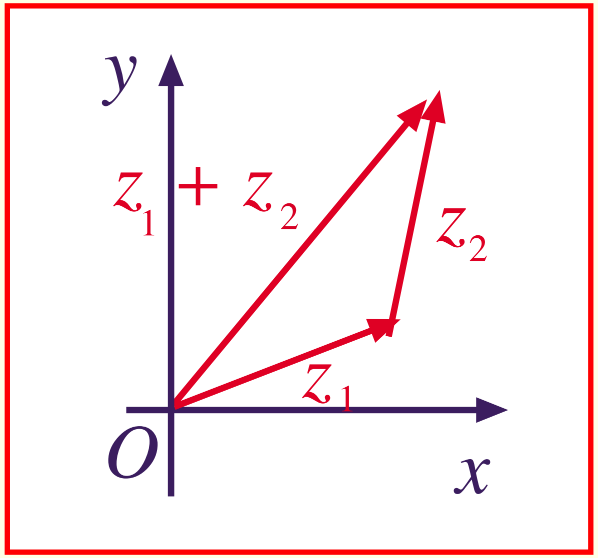
since $z_1\bar{z}_2 + z_2\bar{z}_1 = 2\,\mathrm{Re}(z_1\bar{z}_2) \le |z_1\bar{z}_2|$

$$= |z_1.z_2| = |z_1|.|z_2|;$$

$$= (|z_1| + |z_2|)^2$$

More on the triangle inequality

The triangle inequality becomes clear when seen geometrically, as in the adjacent figure. It just says that the length of any side of a triangle is not more than the sum of the lengths of the remaining two sides.



Another useful result is the **ModDiff Inequality**:

$$\boldsymbol{\big|\,|z_1| - |z_2|\,\big| \le |z_1 + z_2|\,.}$$

We can think of this as providing a lower bound for $|z_1 + z_2|$.

Proof $\quad |z_1| = |(z_1 + z_2) + (-z_2)| \le |z_1 + z_2| + |-z_2|$

so $\qquad |z_1| - |z_2| \le |z_1 + z_2|.$

Similarly,

$$|z_2| - |z_1| \le |z_1 + z_2|,$$

giving the required result.

QUIZ 1.3A

1. $|3+4i| =$ [ ]

2. If $z = \bar{z}$, then z is

(a) Real ✔ ; (b) Imaginary ☐ ;

(c) Neither ☐ .

3. $\overline{z_1 z_2} = \overline{z_1} . \overline{z_2}$.

(a) True ✔ ; (b) False ☐ .

4. $|z_1 + z_2| \; ? \; |z_1| + |z_2|$.

(a) $\geq$ ✔ ; (b) $\leq$ ☐ .

CHECK

QUIZ 1.3B

The Triangle Inequality is $|z_1 + z_2| \le |z_1| + |z_2|$.

Proof: $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = \{\ \mathbf{1}\ \}$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \le \{\ \mathbf{2}\ \}$

(since $z_1\bar{z}_2 + z_2\bar{z}_1 = 2\{\ \mathbf{3}\ \} \le 2|z_1\bar{z}_2| = 2|z_1 z_2| = \{\ \mathbf{4}\ \}$.)

$= (|z_1| + |z_2|)^2$

Match the above boxes **1**, **2**, **3**, **4** with the selections

(a) $\mathrm{Re}\,(z_1\bar{z}_2)$, **(b)** $2|z_1|.|z_2|$,

(c) $|z_1|^2 + 2|z_1|.|z_2| + |z_2|^2$, **(d)** $(z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

1. [ ] 2. [ ] 3. [ ] 4. [ ]

CHECK

Polar Coordinates

We say that $z = (x, y)$ has **polar coordinates** (r, θ) when $x = r\cos\theta, y = r\sin\theta$. We write

$$z = r\cos\theta + ir\sin\theta = r\,\mathrm{cis}\,\theta.$$

Example

$$1 + \sqrt{3}\,i = 2(\cos {}^{\pi}/_{3} + i\sin {}^{\pi}/_{3}).$$

Note that $r = |z|$. We write $\theta =$ **arg** z – the **argument** of z. The argument is always expressed in radians. We can find θ from the relationship $\tan\theta = {}^{y}/_{x}$.

The angle θ is not unique: thus $\theta \pm 2k\pi$ is equally valid.

We use **Arg** z to denote the value of arg z satisfying $-\pi < \arg z \leq \pi$.
This is the **principal value** of the argument.

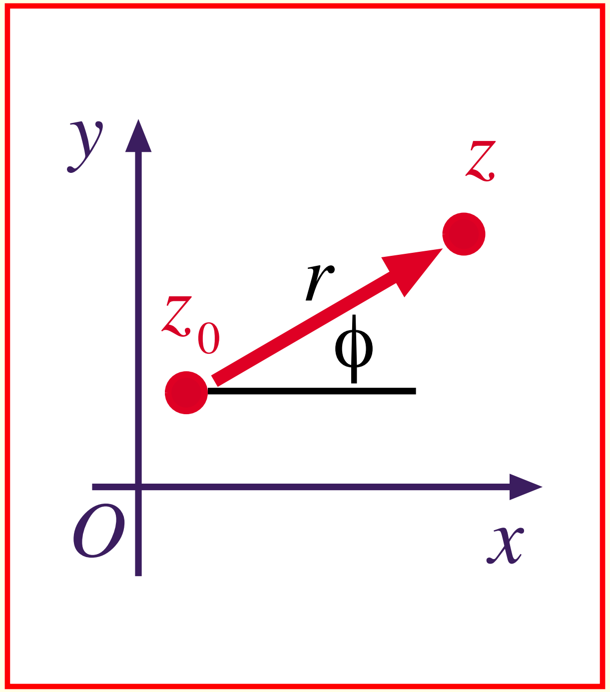
If $z = 0$, arg z is undefined. We adopt the convention that if z is expressed in polar form, then $z \neq 0$.

Using polar coordinates

We begin with a natural extension: $z - z_0 = \text{cis}\,\theta$.

Here, $r = |z - z_0|$, and $\theta = \arg(z - z_0)$.

The adjacent figure illustrates the situation.



The polar form is important because it gives a very simple method of multiplying and dividing complex numbers. This is because

$$\mathbf{arg}\,(z_1 \,.\, z_2) = \mathbf{arg}\, z_1 + \mathbf{arg}\, z_2 \qquad (*)$$

We shall prove this shortly, but notice that this relation may fail for the principal value 'Arg'.

Example

Set $\quad z_1 = -1,\ z_2 = i$.

Then $\quad \text{Arg}\,(z_1 \,.\, z_2) = \text{Arg}\,(-i) = -\pi/2$, while

$\quad \text{Arg}\, z_1 + \text{Arg}\, z_2 = +\pi + \pi/2 = 3\pi/2$.

Proof of the Argument Identity

We now prove the identity $\mathbf{arg}\ (z_1 . z_2) = \mathbf{arg}\ z_1 + \mathbf{arg}\ z_2$ (*):

Set $z_1 = r_1 \operatorname{cis} \theta_1$, $z_2 = r_2 \operatorname{cis} \theta_2$.

Now

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\
&= r_1 r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\
&= r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \\
&= r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)
\end{aligned}
$$

So any argument of z_1 plus any argument of z_2 is an argument of $z_1 z_2$.

On the other hand, if $\arg (z_1 z_2) = \theta_1 + \theta_2 + 2k\pi$, then for example, we could take $\arg z_1 = \theta_1$, and $\arg z_2 = \theta_2 + 2k\pi$.

Euler's Formula

Euler's Formula is:

$$\boldsymbol{e^{i\theta} = \text{cis}\,\theta = \cos\theta + i\sin\theta\,.}$$

We will justify this later. But note that

$$e^{i\theta}\,.\,e^{i\phi} = e^{i(\theta+\phi)}$$

corresponding to

$$\text{cis}\,\theta\,.\,\text{cis}\,\phi = \text{cis}\,(\theta+\phi).$$

In particular,

$$z^{-1} = \tfrac{1}{r}e^{-\theta} = \tfrac{1}{r}\,\text{cis}(-\theta),$$

$$z_1.z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)} = r_1 r_2\,\text{cis}(\theta_1+\theta_2),$$

$$z_1/z_2 = r_1/r_2\, e^{i(\theta_1-\theta_2)} = r_1/r_2\,\text{cis}(\theta_1-\theta_2).$$

The j - operator

We can use complex numbers to represent geometric transformations in the plane, and we shall develop this idea later.

But as an example, consider the mapping

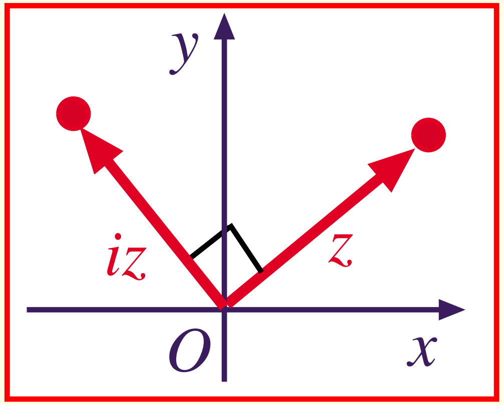
$$z \to f(z) = iz.$$

Here we have $|i| = 1$, $\arg i = \pi/2$.

So $\quad z = r\,\text{cis}\,\theta \;\to\; iz = r\,\text{cis}(\theta + \pi/2)$.

That is, multiplication of z by i effects a rotation through $\pi/2$.

This is the Engineers' ***j*** **- operator**.



"The number you are ringing is imaginary," said the operator. "Please rotate your telephone through 90° and try again!"

Powers and roots

By induction, it is easy to obtain

$$e^{i\theta}.e^{i\theta}. \ldots e^{i\theta} = e^{in\theta},$$

where there are n factors on the left.

Equivalently, we have the well-known de Moivre's Theorem:

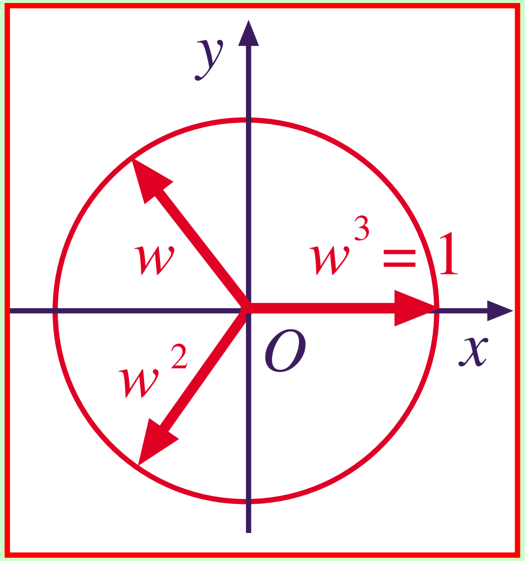
$$(\text{cis}\,\theta)^n = \text{cis}\, n\theta\,.$$

It follows that

$$z^n = r^n \text{cis}\, n\theta = r^n e^{in\theta}.$$

(Feel free to work with either form.)

Example



Solve $z^n = 1$.

Since r^n cis $n\theta = 1$, we have

$$r = 1, \cos n\theta = 1, \sin n\theta = 0$$
$$\Rightarrow \quad n\theta = 2k\pi.$$

The distinct solutions are given by

$$z = \text{cis}\,({}^{2k\pi}/_{n}),\; k = 0, 1, \ldots, n-1.$$

These are called ***n*th roots of unity**.

If $w_n = \text{cis}\,(2\pi / n)$, the n roots are

$$1,\; w_n,\; w_n^{\,2}, \ldots,\; w_n^{\,n-1}.\quad \text{Note that } w_n^{\,n} = 1.$$

The cube roots of 1 occur as the vertices of an equilateral triangle on the unit circle.

General n th roots

We can easily extend the above method to finding n th roots of any $w = \rho \operatorname{cis} \phi$.

For if $z = r \operatorname{cis} \theta$ and $z^n = w$, then we must have $r^n = \rho$, and $n\theta = \phi + 2k\pi$ for integral k.

Thus $r = \sqrt[n]{\rho}$ (the positive nth root), and $\theta = {}^{\phi}/_{n} + {}^{2k\pi}/_{n}$, $k = 0, 1, \ldots, n-1$.

Example

Find all values of $\sqrt[4]{1}$.

We set $z = r \operatorname{cis} \theta$, and $z^4 = 1 = 1.\operatorname{cis} 0$.

Then $\quad r^4 = 1$ implies that $r = 1$,

and $\quad 4\theta = 0 + 2k\pi$ implies that $\theta = {}^{2k\pi}/_{4}$, $k = 0, 1, \ldots, 3$.

Hence the four values of $\sqrt[4]{1}$ are

$1.\operatorname{cis} 0 = 1$, $1.\operatorname{cis} {}^{2\pi}/_{4} = i$, $1.\operatorname{cis} {}^{4\pi}/_{4} = -1$, and $1.\operatorname{cis} {}^{6\pi}/_{4} = -i$.

QUIZ 1.4

1. In polar form, $z \;=\; 1 + \sqrt{3}\, i$

$=$ ☐

2. If $z \;=\; \sqrt{2}\ \mathrm{cis}\ {}^{5\pi}/_{4}$,

then $z \;=$ ☐

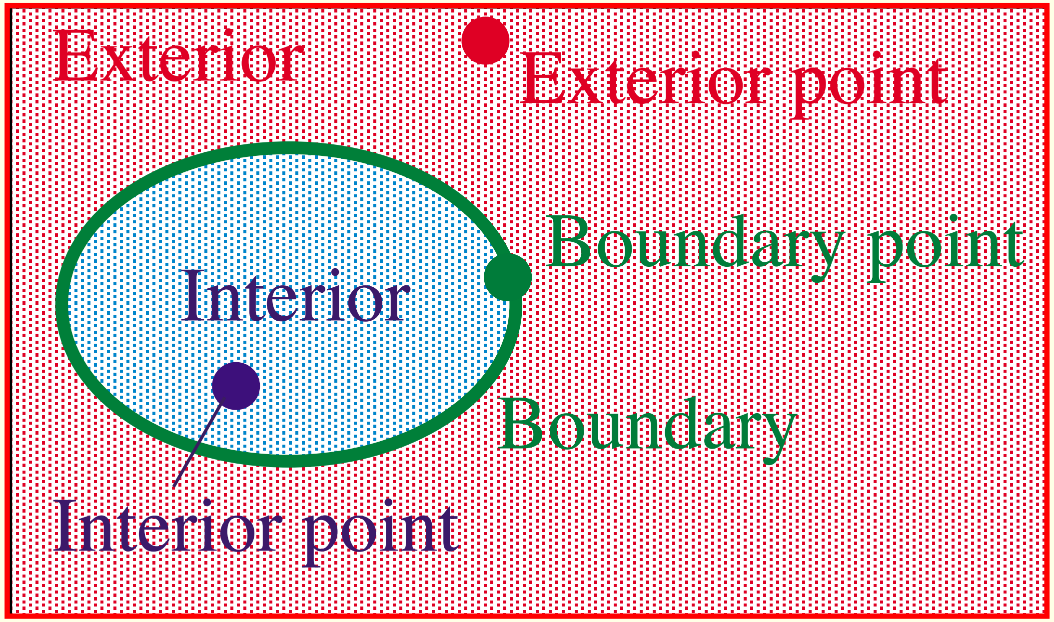
3. A square root of $-i$ is ☐

4. Geometrically, the powers of cis ${}^{\pi}/_{5}$ generate all the vertices of a regular pentagon.

(a) True ✔ ; (b) False ☐ .

CHECK

Regions in the complex plane



We have already mentioned the complex plane. We look at some terms which describe certain sets in the plane.

A **neighbourhood** or more specifically, an **ε - neighbourhood** (epsilon-neighbourhood) of z_0, is the set of points satisfying $|z - z_0| < \varepsilon$. That is, it is the set of points lying inside, but not on, the circle of radius ε centred at z_0.

A point z is said to be **interior** to set S if there is some neighbourhood of z which just contains points of S. A point z is said to be **exterior** to set S if there is some neighbourhood of z which contains no points of S. A point z is said to be a **boundary point** of set S if every neighbourhood of z contains points in S and points not in S. All the boundary points together make up the **boundary** of S. Notice that a boundary point of S need not be an actual point of S.

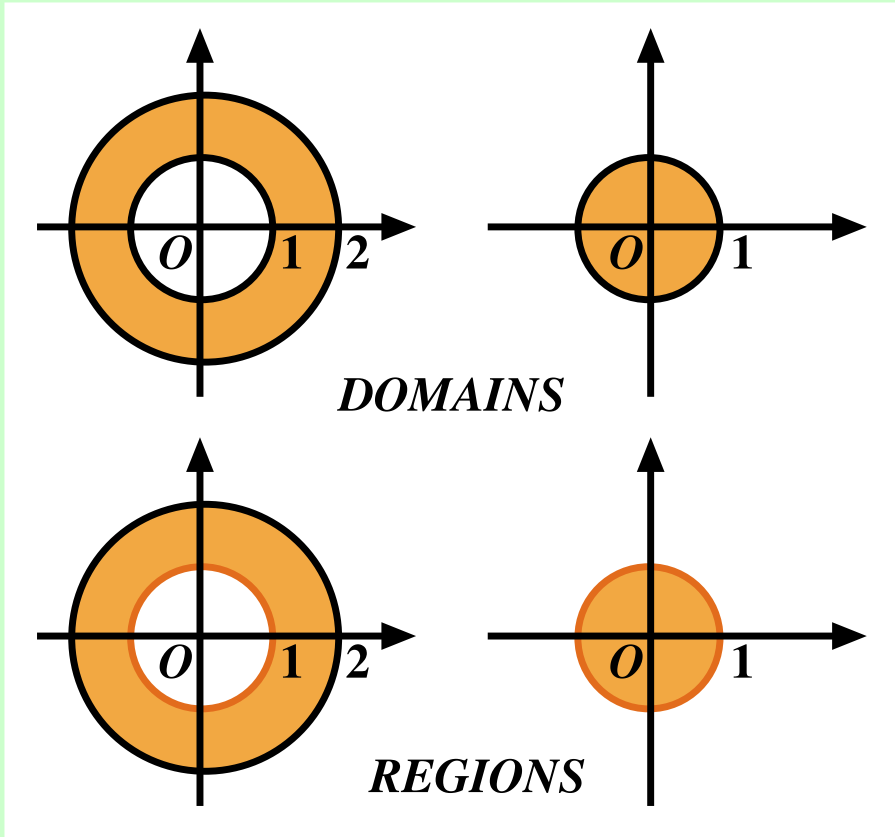
Domains and regions

An open, connected set is called a **domain**. A connected set is called a **region**.

A set S is **bounded** if every point of S lies inside some circle $|z| = R$; otherwise it is **unbounded**.

Examples

The annulus $1 < |z| < 2$ and the disk $|z| < 1$ are domains.
The annulus $1 \leq |z| < 2$ and the disk $|z| \leq 1$ are regions.



Examples

The annulus $1 < |z| < 2$ is bounded (it lies inside the circle $|z| = 3$).
The straight line $\{z = x + iy \mid y = 0\}$ is unbounded.

Points of accumulation

We say that point z_0 is an **accumulation point** of a set S, if each neighbourhood of z_0 contains at least one point of S distinct from z_0.

This is a more difficult concept, and is obviously associated with a limiting process. Thus we would expect to be able to find a sequence of points of S lying closer and closer to z_0.

Lemma ***S* is closed ⇒ *S* contains all its points of accumulation.**

Proof (⇒) Let S be closed. Then S contains all its interior and boundary points. A point of accumulation of S can not be an exterior point (see the definition of exterior point). Hence S contains all its points of accumulation.

(⇐) Suppose now that S contains all its points of accumulation. Does S contain all its boundary points? Let z_0 be a boundary point of S not in S. By definition of boundary point, each neighbourhood of z_0 contains a point of S, so z_0 is a point of accumulation of $\overline{S}$. So z_0 lies in S and S contains all its boundary points. Hence S is closed.

QUIZ 1.5

1. The set $2 \le |z| < 3$ is open / closed / neither?

2. The set $|z| < 3$ is a domain because it is

(a) open ✔ ; (b) connected ☐ ;

(c) both of these ☐ .

3. The point $z = 1$ is an accumulation point of $|z| < 1$.

(a) True ☐ ; (b) False ✔ .

4. A discrete point set can have a point of accumulation.

(a) True ✔ ; (b) False ☐ .

CHECK