2. ANALYTIC FUNCTIONS

Functions of a complex variable

Let D be a subset of C. A **function** $f : D \to C$ is a rule that associates with each z in D a unique complex number w. We write $w = f(z)$.

Notes

1. The set D of numbers that are mapped is called the **domain** of f. Notice that we now have a double use of this word. Where the domain is unspecified, we assume it to be the largest subset of C for which $f(z)$ is defined.

2. The set of image elements $\{w \mid w = f(z)\}$ is called the **range** or **image** of the function.

3. The above definition specifies a *unique* image for each $z \in D$. Later we shall extend this definition to include *multivalued* functions.

Examples of functions

If $f(z)$ only assumes real values, f is **real-valued**.

Example 1

$$f_1(z) = z^2 + iz - 3.$$

The domain is $\mathbf{C}$.

$$f_2(z) = {}^1/_{(1 + z^2)}.$$

Here the domain is $\mathbf{C} \setminus \{-i\}$.

Example 2

$$f_3(z) = |z|,$$

$$f_4(z) = \operatorname{Re} z = x,$$

$$f_5(z) = \operatorname{Im} z = y.$$

We can break $w = f(z)$ into real and imaginary parts. Thus if $w = u + iv, z = x + iy,$ then

$$w = f(z) = u(x, y) + iv(x, y).$$

Example 3

$$f_6(z) = z^2 .$$

Here $u + iv = (x + iy)^2 = (x^2 - y^2) + 2ixy,$

so

$$u = (x^2 - y^2), \ v = 2xy.$$

In practice, this expression in terms of real and imaginary parts may be easier said than done! In theory, it allows us to deduce properties of complex functions from our knowledge of the real numbers.

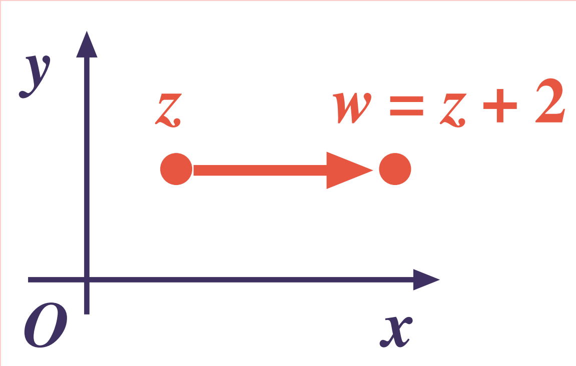
Mappings

A real function $y = f(x)$ can be represented geometrically by a graph: the set of points $\{(x, y) \mid y = f(x)\}$. To represent the complex function $w = f(z)$ geometrically, in general we need four dimensions or two planes: a plane for the domain, and a plane for the range. For simple functions, we can use the same plane twice.

Example 1

$w = f(z) = z + 2.$

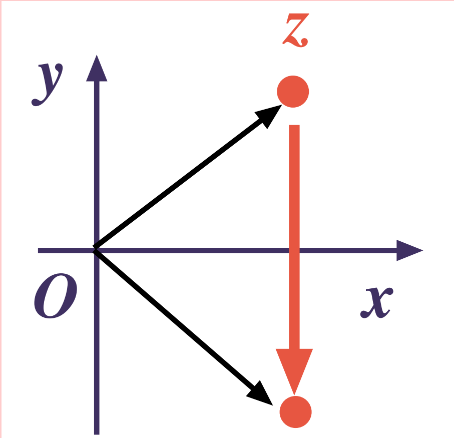
This is a translation: each point z is translated through 2 to the point $z + 2$.



Example 2

$w = f(z) = \bar{z}.$

This is a reflection in the x-axis: each point $z = x + iy$ (or (x, y)) is mapped to the point $\bar{z} = x - iy$ (or $(x, -y)$).



QUIZ 2.1

1. The domain of $f = f(z)$ is the set of image elements.
(a) True ☐ ; (b) False ☐ .

2. If $z = x + iy$, then the function $f(z) = y$ is real valued.
(a) True ☐ ; (b) False ☐ .

3. Describe geometrically the mapping
$f(z) = -z.$

4. Describe geometrically the mapping
$f(z) = 2z.$

CHECK

Mapping curves and regions

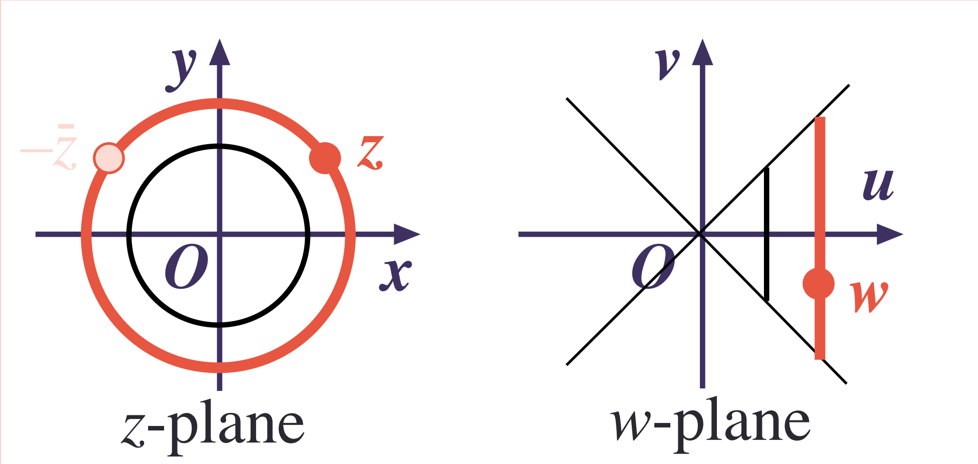
We shall be mapping *curves* and *regions* rather than just points.

Example

Find the image of the circle

$$x^2 + y^2 = c^2 \ (c > 0)$$

under $w = f(z) = \sqrt{(x^2 + y^2)} - iy$.



Let us set $w = u + iv$. Then each point (x, y) on the circle $x^2 + y^2 = c^2$ maps to $(u, v) = (c, -y)$, where $|y| \le c$.

Thus the image of this circle is the line segment $u = c, \ -c \le v \le c$ in the uv-plane. The domain of f is the z-plane; the range of f is a quadrant of the w-plane.

Notice that $z = (x, y)$ and $-\bar{z} = (-x, y)$ map to the same point w.

Limits

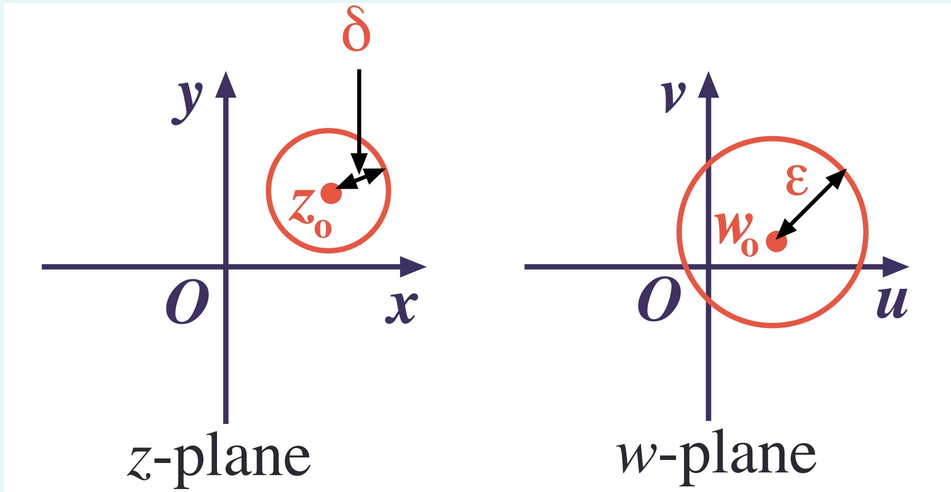
All work on functions of two variables now carries over directly. A minor difference is that because we are dealing with complex numbers, the length or norm of a vector represented by w becomes the modulus $|w|$ of w.

Thus the definition of limit becomes:

$\lim_{z \to z_0} f(z) = w_0$ means

for all $\varepsilon > 0$, there exists $\delta > 0$: for all z, $0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$.

Thus every z in the left disc has an image in the right disk. We may not obtain the whole of the right disc; for example, consider the image of $f(z)$ = constant = w_0; we obtain just the central point of the second disk.



A limit theorem (I)

Theorem 2.1 If $w = f(z) = u + iv$, $z = x + iy$, $z_0 = x_0 + iy_0$, then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0 \quad \Leftrightarrow \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \text{ and } \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0 .$$

In brief, this theorem says: $\lim(u + iv) = \lim u + i \lim v$.

Proof (⇒) Let us suppose that

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

By definition, given ε, there exists $\delta > 0$:

$$0 < |x - x_0 + i(y - y_0)| < \delta$$

$$\Rightarrow |u(x, y) - u_0 + i(v(x, y) - v_0)| < \varepsilon.$$

We deduce that

$$0 < |x - x_0| < {}^{\delta}/_{2}, \ 0 < |y - y_0| < {}^{\delta}/_{2}$$

$$\Rightarrow |u(x, y) - u_0| < \varepsilon, \ |v(x, y) - v_0| < \varepsilon.$$

This completes this part of the proof.

A limit theorem (II)

Theorem 2.1 If $w = f(z) = u + iv$, $z = x + iy$, $z_0 = x_0 + iy_0$, then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0 \quad \Leftrightarrow \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \text{ and } \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0 .$$

(⇐) Now let us suppose that

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \text{ and } \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0 .$$

Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$0 < |x - x_0| < \delta_1,\ 0 < |y - y_0| < \delta_1 \Rightarrow |u(x, y) - u_0| < {}^{\varepsilon}/_{2},$$

$$0 < |x - x_0| < \delta_2,\ 0 < |y - y_0| < \delta_2 \Rightarrow |v(x, y) - v_0| < {}^{\varepsilon}/_{2}.$$

Choose $\delta = \min(\delta_1, \delta_2)$. Then using the given limits,

$$0 < |(x - x_0) + i(y - y_0)| < \delta \Rightarrow$$

$$|(u(x, y) + iv(x, y)) - (u_0 + iv_0)| \le |u(x, y) - u_0| + |i||v(x, y) - v_0| < {}^{\varepsilon}/_{2} + {}^{\varepsilon}/_{2} < \varepsilon,$$

as required.

More Limit Theorems

Our previous theorem quickly leads to the well-known and useful **Limit Theorems**. We use an easy to remember abbreviated notation.

Theorem 2.2. (Limit Theorems) If $\lim f$, $\lim g$ exist, then

$$\lim (f \pm g) = \lim f \pm \lim g,$$

$$\lim (f \, . \, g) = \lim f \, . \lim g,$$

$$\lim (f / g) = \lim f / \lim g \qquad (\lim g \neq 0).$$

Proof **(a)** Set $f = u + iv$, $\lim f = u_0 + iv_0$, $g = U + iV$, $\lim g = U_0 + iV_0$.

Now

$$\begin{aligned}
\lim (f + g) &= \lim (u + U + i(v + V)) && \text{(substitute and rearrange)} \\
&= \lim (u + U) + i \lim(v + V) && \text{(Thm 2.1)} \\
&= u_0 + U_0 + i(v_0 + V_0) && \text{(put in the limits)} \\
&= (u_0 + i v_0) + (U_0 + iV_0) && \text{(rearrange)} \\
&= \lim f + \lim g.
\end{aligned}$$

The other proofs are similar.

QUIZ 2:2A

1. Geometrically, the image of $f(z) = 2 + i$ is a

2. Geometrically, mapping $f(z) = \overline{z}$ maps the square with vertices $(\pm 1, \pm i)$ onto itself.

(a) True ☐ ; (b) False ☐ .

3. If $u + iv$ tends to $u_0 + iv_0$, then we must have $u \to u_0$.

(a) True ✔ ; (b) False ☐ .

4. If $f(z) = 2 + i$ and $g(z) = 2 - i$ then $(f.g)(z) =$

CHECK

QUIZ 2:2B

Theorem 2.1 $\lim_{z \to z_0} f(z) = u_0 + iv_0 \iff \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0$ and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0$.

Proof (⇒) Let us suppose that **{ 1 }**. By definition, given ε, there exists $\delta > 0$:

$$0 < \{ 2 \} \Rightarrow |u(x, y) - u_0 + i(v(x, y) - v_0)| < \varepsilon.$$

We deduce that

$$\{ 3 \}, \quad 0 < |y - y_0| < \delta/2$$
$$\Rightarrow |u(x, y) - u_0| < \varepsilon, \{ 4 \}.$$

This completes this part of the proof.

1	**(c)**
2	**(b)**
3	**(d)**
4	**(a)**

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Match the above numbers **1, 2, 3, 4** with the selections:

(a) $|v(x, y) - v_0| < \varepsilon$ **(b)** $|(x - x_0) + i(y - y_0)| < \delta$

(c) $\lim_{z \to z_0} f(z) = u_0 + iv_0$ **(d)** $0 < |x - x_0| < \delta/2$

1. ______ **2.** ______ **3.** ______ **4.** ______

Continuity

Definition Function f is said to be **continuous** at z_0 if f satisfies the following three conditions.

(a) $f(z_0)$ exists ; (b) $\lim_{z \to z_0} f(z)$ exists ; (c) $\lim_{z \to z_0} f(z) = f(z_0)$.

Notes

1. Just writing statement (c) assumes the truth of (a) and (b).

2. Expressing (c) in terms of the limit definition, we obtain:

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

This is slightly different from the usual definition of limit, in that we allow the possibility $z = z_0$ (omitting $0 < |z - z_0| \ldots$).

Function $f = f(z)$ is said to be **continuous in a region** if it is continuous at each point of the region.

Propertiers of continuous functions

The sum $f + g$, difference $f - g$, product $f.g$ and quotient f/g of two continuous functions is continuous at a point $z = z_0$, with the proviso that in the last case $g(z_0) \neq 0$. These results follow directly from the Limit Theorems 2.2.

Examples

1. Polynomial functions

The polynomial function $P(z) = \Sigma a_i z^i$ is continuous for all z since it is constructed as sums and products of the continuous constant functions (a_i) and the continuous function $f(z) = z$.

2. Rational functions

The rational function $^{P(z)}/_{Q(z)}$ given by the quotient of two continuous polynomial functions $P(z), Q(z)$ is continuous for all $z : Q(z) \neq 0$.

Composition of continuous functions

We can also compose complex functions f, g to obtain the new function $f \circ g$ defined by $(f \circ g)(z) = f(g(z))$. If f, g are continuous, will $f \circ g$ be continuous also?

Theorem 2.3 The composite function $f \circ g$ of two continuous functions f, g is continuous.

Alternatively, a continuous function of a continuous function is a continuous function.

Formally, the proof of this theorem is exactly as for the real case, and is omitted here.

Example $f(z) = \sin(z^2)$ is continuous for all z.

By Theorem 2.1, $f(z) = u + iv$ is continuous $\Leftrightarrow$ $u(x, y), v(x, y)$ are continuous. Thus:

Example $f(z) = e^{xy} + i\sin(x^2 - 2yx^3)$ is continuous for all z (since the real and imaginary parts are continuous).

Definition We say f is **bounded** in region R if $|f(z)| \leq M$ for all $z \in R$.

If f is continuous in R, then f is bounded because of the corresponding properties of u, v.

Show this!

Quiz 2.3

1. If $f(z_0)$ exists, then function f must be continuous at $z = z_0$.

(a) True ; (b) False .

2. If $\lim_{z \to z_0} f(z)$ exists, then function f must be continuous at $z = z_0$.

(a) True ; (b) False .

3. The function $f(z) = \sin\left({}^{1}/_{z}\right)$ is continuous everywhere.

(a) True ; (b) False .

4. The function $f(z) = \cos(z^3)$ is continuous everywhere.

(a) True ; (b) False .

The derivative

Formally, the definition of the derivative ${}^{df}/_{dz} = f'(z)$ for functions of a complex variable is the same as for real functions.

Let f be a function whose domain contains a neighbourhood of point z_0. Then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if the limit exists. In this case the function f is said to be **differentiable** at z_0.

It is sometimes preferable to use the alternative form of the derivative obtained by setting $z = z_0 + \Delta z$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Note Since z_0 lies in an (open) neighbourhood of the domain of f, $f(z_0 + \Delta z)$ is defined if Δz is small.

Examples of the derivative

We can evaluate simple derivatives by using the basic definition.

Example $f(z) = z^2$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z.\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z.$$

The usual differentiation formulae hold as for real variables.

For example, $\frac{d}{dz}(c) = 0$, $\frac{d}{dz}(z) = 1$, $\frac{d}{dz}(z^n) = n z^{n-1}$.

However, care is required for more unusual functions.

Example $f(x) = |x|^2 = x^2 \Rightarrow f'(x) = 2x$ for all x.

But $f(z) = |z|^2 \Rightarrow f'(z)$ exists only at $z = 0$.

This last statement is proved using the basic definition. Show it!

More on the derivative

As in the real case, f is differentiable $\Rightarrow$ f is continuous.

The same rules apply in the complex case for the sum, product, quotient and composite of two differentiable functions (where defined).

Example $\frac{d}{dz}(2z^2 + i)^5 = 5(2z^2 + i)^4.4z = 20z(2z^2 + i)^4.$

Again, with more unusual functions, we may have to use the limit definition of differentiation.

Example Investigate $\frac{d}{dz}(\text{Re } z)$.

We get $\lim_{z \to z_0} \frac{\text{Re}(z) - \text{Re}(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{x - x_0}{(x - x_0) + (y - y_0)}$

Here we get 0 approaching along $x = x_0$, 1 along $y = y_0$. Hence the limit does not exist.

Quiz 2.4

1. If function f is continuous at $z = z_0$, then f must be differentiable there.

(a) True ; (b) False .

2. If $f(z) = |z|^2$, then for all z,

$f'(z) =$ ______

3. If $f(z) = (iz + 2)^2$, then

$f'(z) =$ ______

4. If $f(z) = \cos(z^3)$, then

$f'(z) =$ ______

1. False.
The converse is true

2. $f'(z) = 0$ if $z = 0$.
Else $f'(z)$ does not exist.

3. $2i(iz + 2)$. Expand and differentiate, or directly.

4. $-\sin(z^3).\, 3z^2$.
Use the Chain Rule.

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Cauchy-Riemann Equations (I)

Theorem 2.1 gives us conditions for continuity for a function of a complex variable in terms of the continuity of the real and imaginary parts. We now ask: Is there any test for differentiability?

Theorem 2.4 The derivative $f'(z)$ of $f = u + iv$ exists at z $\Leftrightarrow$ the first order partial derivatives u_x, v_x, u_y, v_y all exist and satisfy

$$u_x = v_y, \quad u_y = -v_x \qquad \text{(the **Cauchy-Riemann equations**)}.$$

Further, $$f'(z) = u_x + iv_x = v_y - iu_y.$$

Proof Since the derivative of f exists,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{u(x, y) + iv(x, y) - u(x_0, y_0) + iv(x_0, y_0)}{x + iy - x_0 - iy_0} \qquad (*)$$

$$= a + ib \quad \text{(say)}.$$

(continued ...)

Cauchy-Riemann Equations (II)

By Theorem 2.1, the limit of the real part of (*) = a, the limit of the imaginary part of (*) = b.

Set $y \equiv y_0$ to get $$\lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} = \frac{\partial u}{\partial x} = a$$

and $$\lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} = \frac{\partial v}{\partial x} = b$$

Set $x \equiv x_0$ to get $$\lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} = \frac{\partial v}{\partial y} = a$$

and $$\lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} = \frac{\partial u}{\partial y} = -b$$

Hence all the first partial derivatives exist, $u_x = v_y$, $u_y = -v_x$, and

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$ etc. as required.

Cauchy-Riemann Examples

1. Set $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$.

Now $f'(z) = 2z$ exists for all z. So the Cauchy-Riemann equations are satisfied. We have $u = x^2 - y^2$, $v = 2xy$, and $u_x = 2x = v_y$, $u_y = -2y = -v_x$.

Also $f'(z) = u_x + iv_x = 2x + 2iy = 2z$ as expected.

2. Set $f(z) = |z|^2$. We show $f'(z)$ does not exist for $z \neq 0$.

Now $f(z) = x^2 + y^2$, i.e. $u = x^2 + y^2$, $v = 0$, $u_x = 2x$, $u_y = 2y$, $v_x = 0 = v_y$.
So $u_x = v_y \Rightarrow x = 0$, $u_y = -v_x \Rightarrow \mathrm{y} = 0$.
Hence $f'(z)$ can only exist at $(0, 0)$.

Does $f'(0)$ exist? Yes; as suggested earlier, but we must use a first principles argument to show it.

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Sufficient conditions

Theorem 2.4 gives *necessary* conditions for f to be differentiable. We now seek *sufficient* conditions for f' to exist: that is, a similar statement to Theorem 2.4, but using $\Leftarrow$.

Theorem 2.5 Let $f = u + iv$ as before. Suppose

(i) u, v, u_x, v_x, u_y, v_y exist in the neighbourhood of (x_0, y_0),

(ii) u_x, v_x, u_y, v_y are continuous at (x_0, y_0),

(iii) the Cauchy-Riemann equations are satisfied at (x_0, y_0).

Then $f'(z)$ exists and at z_0, $f'(z_0) = u_x + iv_x$ as before. That is,

f is differentiable at z_0 $\Leftarrow$ the given conditions.

Proof We omit this proof. It is not hard, but rather messy.

Example

The real functions $u = e^x \cos y$, $v = e^x \sin y$ are defined and continuous everywhere. So are u_x, v_x, u_y, v_y and you can easily check that the Cauchy-Riemann equations are satisfied.

Hence the function

$$f(z) = e^x \cos y + i e^x \sin y$$

is differentiable everywhere.

Since $u_x = u$, $v_x = v$,

$$f'(z) = f(z)$$

$$(= e^x \operatorname{cis} y = e^{x+iy} = e^z).$$

Quiz 2.5

1. If $f(z) = u + iv$ and the Cauchy-Riemann equations hold for u, v, then $f'(z)$ must exist.

(a) True ☐ ; (b) False ✔ .

2. For $f = u + iv$, the Cauchy-Riemann equations are $u_x = v_y$ and $v_x = u_y$.

(a) True ☐ ; (b) False ✔ .

3. If $f(z) = (x^2 - y^2 + 2) + 2ixy = u + iv$, then the Cauchy-Riemann equations hold.

(a) True ✔ ; (b) False ☐ .

4. If $f(z)$ is differentiable, then $f'(z) = v_y - i u_y$.

(a) True ☐ ; (b) False ☐ .

CHECK

Analytic functions

Definitions Function $f(z)$ is **analytic at z_0** if $f'(z)$ exists not only at z_0 but for all z in some neighbourhood of z_0. $f(z)$ is **analytic in a domain** of the z-plane if it is analytic at every point of the domain. $f(z)$ is **entire** if it is analytic everywhere.

Examples

1. $f(z) = |z|^2$ is not analytic anywhere. (It is in fact differentiable only at $z = 0$).

2. $f(z) = {}^1/_z$ is analytic (except at $z = 0$).

3. $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ is entire.

If $f(z)$ is analytic throughout a domain except for a finite number of points, such points are **singularities** or **singular points** of f.

Examples

$f(z) = \dfrac{1}{z}$ ($z = 0$ is a singularity); $f(z) = \dfrac{1}{(z-1)(z-2)}$ ($z = 1, 2$ are singularities).

Test for analytic functions

Question How can we tell if a function is analytic? We can use Theorem 2.5, or

Theorem 2.6 If $f = f(z)$ is analytic, then in any formula for f, x and y can only occur in the combination $x + iy$.

Proof We note that $x = \frac{1}{2}(z + \bar{z})$, $y = (\frac{1}{2i})(z - \bar{z})$. Hence if $w = f(z) = u + iv$, we can regard u, v as functions of $z, \bar{z}$. Now, w is a function of z alone $\Leftrightarrow \frac{\partial w}{\partial \bar{z}} = 0$, and

$$\frac{\partial w}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) = 0$$

$$\Leftrightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial y} = 0$$

$$\Leftrightarrow u_x = v_y, \quad u_y = -v_x,$$

equating real, imaginary parts to zero.

Hence f analytic $\Rightarrow$ the Cauchy-Riemann equations hold $\Rightarrow \frac{\partial w}{\partial \bar{z}} = 0$ as required.

Analytic functions : final comments

Example

$$f(z) = \sin(x + 3iy)$$

We can say immediately that this function is not analytic, as x and y do not occur in the combination $x + iy$. In some examples it is less clear whether or not the variables can be combined in this way.

Derivative theorems

The theorems on derivatives quickly transfer to analytic functions. Thus the sum, product, quotient and composite of two analytic functions are formed in the obvious ways as before, and each of the resulting functions is analytic on its domain.

Augustin -Louis Cauchy

The name of Cauchy [pronounced ‘Co′-shee’] (1789 – 1857) is found frequently in complex analysis. This is because over much of his life, he almost single-handedly developed the theory of complex functions. He had a prodigious output, writing several books and 789 papers, some of great length.

Harmonic functions

Let $f = u + iv$ be analytic in some domain of the z-plane. Then the Cauchy-Riemann equations hold:

$$u_x = v_y, \quad u_y = -v_x.$$

It can be shown that for analytic function, the partial derivatives of all orders exist and are continuous functions of x, y. Hence

$$u_{xx} = v_{yx} \quad \text{and} \quad u_{yy} = -v_{xy}.$$

Assuming continuity of v_{yx}, v_{xy}, we have $v_{yx} = v_{xy}$, and hence

$$\boldsymbol{u_{xx} + u_{yy} = 0}.$$

This is **Laplace's equation**, and u is called an **harmonic function**.

In the same way we get

$$v_{xx} + v_{yy} = 0; \quad \text{i.e. } v \text{ is an harmonic function.}$$

If $f = u + iv$, u and v are **conjugate harmonic functions**.

(Note the different use of the word 'conjugate' here).

Finding harmonic functions

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In applied mathematics (partial differential equations) we often seek an harmonic function on a given domain which satisfies given boundary conditions. If we are given one of two conjugate harmonic functions, it is a simple matter to find the other. We use the Cauchy-Riemann equations.

Example

Let $u = y^3 - 3x^2y$. Then u is harmonic, since $u_{xx} = -6y = -u_{yy}$.

Now using the Cauchy-Riemann equations, $u_x = -6xy = v_y$.

Integrating v partially with respect to y gives $v = -3xy^2 + \phi(x)$ and now

$$v_x = -u_y = -3y^2 + 3x^2 \Rightarrow \phi'(x) = 3x^2.$$

Hence $v = -3xy^2 + x^3 + c$.

You can check that v is in fact harmonic! So

$$f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2 + c) \quad [= i(z^3 + c) \text{ in fact }].$$

Quiz 2.6A

1. If $f(z)$ is analytic, then $f'(z)$ exists.

(a) True ☐ ; (b) False ☐ .

2. Function $f(z)$ may be differentiable at $z = z_0$, but not analytic near $z = z_0$.

(a) True ☐ ; (b) False ☐ .

3. Function $v(x, y) = -3xy^2 + x^3$ is an harmonic function.

(a) True ☐ ; (b) False ☐ .

4. The harmonic conjugate of $u(x, y) = -2xy$ is

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Quiz 2.6B

Theorem 2.6 If $f = f(z)$ is analytic, then in any formula for f, x and y can only occur in the combination $x + iy$.

Proof We note that $x = \frac{1}{2}(z + \bar{z})$, **{1}** . Hence if $w = f(z) = u + iv$, we can regard u, v as functions of $z, \bar{z}$. Now, w is a function of z alone $\Leftrightarrow$ **{2}** and

$$\frac{\partial w}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} + i\left(\{3\}\right) = 0$$

$$\Leftrightarrow \{4\} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial y} = 0 \Leftrightarrow u_x = v_y, \quad u_y = -v_x,$$

equating real, imaginary parts to zero.

Hence f analytic $\Rightarrow$ the Cauchy-Riemann equations hold

$\Rightarrow \frac{\partial w}{\partial \bar{z}} = 0$ as required.

Match the above boxes 1, 2, 3, 4 with the selections **(a)** $y = (\frac{1}{2i})(z - \bar{z})$, **(b)** $\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}$ **(c)** $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial y}$ **(d)** $\frac{\partial w}{\partial \bar{z}} = 0$,

1. ☐ **2.** ☐ **3.** ☐ **4.** ☐ ©

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