3. ELEMENTARY FUNCTIONS

Exponential and trigonometric functions

If $z = x + iy$, we define the **exponential function** $\mathbf{exp}\ z = e^x \operatorname{cis} y$ $(= e^z)$.

Notes

1. We can give an alternative definition in terms of power series. Writing out a formal series for e^{iy} gives $\operatorname{cis} y$.

2. If $y = 0$, then $\exp z = \exp x = e^x$. Thus the complex exponential function naturally extends the real function.

3. In this definition, y is in radian measure.

Properties of the exponential (I)

1. The function exp is entire and $\frac{d}{dz}(\exp z) = \exp z$.
(**See** Thm 2.5 and the example following it .)

2. If $w = w(z)$ is analytic in some domain D, then so is $\exp w$.

3. The function $\exp z = e^x \operatorname{cis} y$ is a complex number in *polar* form:
$|\exp z| = e^x$, $\arg(\exp z) = y$.

4. The range of $w = \exp z$ is the whole w-plane except O.
For, $w = e^x \operatorname{cis} y$ with $e^x > 0$; to get $w = \rho \operatorname{cis} \phi$, set $x = \ln \rho$ $(\rho > 0)$ and $y = \phi$.

5. Laws of exponents:

$$\exp z_1 . \exp z_2 = \exp(z_1 + z_2) \;; \qquad \exp z_1 / \exp z_2 = \exp(z_1 - z_2)$$

6. Powers:

$$(\exp z)^m = \exp(mz) \quad m \in \mathbf{Z}^+$$

$$(\exp z) = \exp {}^1/_n (z + 2k\pi i) \quad m, n \in \mathbf{Z}^+$$

$$(\exp z)^{m/n} = \exp {}^m/_n (z + 2k\pi i) \quad k \in \mathbf{Z}^+$$

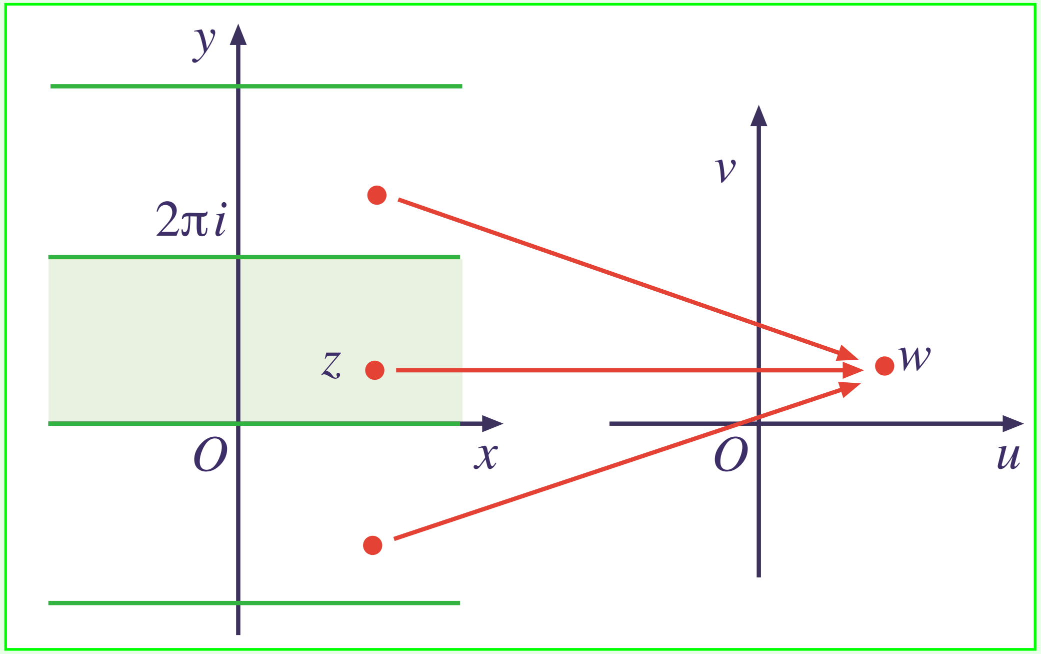
The proofs follow directly from the definition of the exponential. Note that $(e^x \operatorname{cis} y)^{1/n} = e^{x/n} \operatorname{cis}((y + 2k\pi)/n)$.

Properties of the exponential (II)

7. We observe that $\exp(z + 2\pi i) = \exp z . \exp(2\pi i)$,
and that $\exp(2\pi i) = e^0.(\cos 2\pi + i \sin 2\pi) = 1$.

It follows that $\exp(z + 2\pi i) = \exp z$.

Thus we can divide the z-plane into **periodic strips**. Each strip in the z-plane is mapped to the whole w-plane excluding the origin. Thus the exponential function is **periodic** with a period of $2\pi i$.



We note the further two properties of the exponential:

8. $\exp \overline{z} = \overline{\exp z}$.

9. $\text{cis}\,\theta = \cos\theta + i\sin\theta = \exp(i\theta)$.

Quiz 3.1

1. Function $f(z) = 3e^{2z} + 4e^{z}$ is entire.
(a) True ✔ ; (b) False .

2. $f(z) = \exp(3 + \pi i) = e^{3}$.
(a) True ✔ ; (b) False .

3. The range of $w = f(z) = e^{z}$ is the whole complex w-plane.
(a) True ✔ ; (b) False .

4. $\exp(2 + 3i) \,.\, \exp(4 + 5i) = e^{6}\text{cis } 8$.
(a) True ✔ ; (b) False .

5. $|\exp(3i)| = 3$.
(a) True ✔ ; (b) False .

CHECK

Sine and cosine

If y is a real number, we have

$$\exp(iy) = \cos y + i \sin y, \quad \exp(-iy) = \cos y - i \sin y,$$

and so

$$\cos y = \tfrac{1}{2}.(\exp(iy) + \exp(-iy)),$$
$$\sin y = \tfrac{1}{2} i.(\exp(iy) - \exp(-iy)).$$

Thus it is natural to define **cosine** and **sine** as:

$$\cos z = \tfrac{1}{2}\,.\,(\exp(iz) + \exp(-iz)),$$
$$\sin z = \tfrac{1}{2} i.(\exp(iz) - \exp(-iz)).$$

These are **Euler's relations**.

Again notice here how we try to generalize, or extend, a 'real' situation to the complex case.

Properties of sine and cosine

1. Both functions are entire:

$$\frac{d}{dz}(\sin z) = \cos z, \quad \frac{d}{dz}(\cos z) = -\sin z.$$

2. Both functions are periodic, of period 2π.
This follows from the periodicity of the exponential function.

The functions satisfy the usual identities, as in the real case.

3. $\sin^2 z + \cos^2 z = 1.$

4. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$ etc.

5. $\sin(-z) = -\sin z, \quad \cos(-z) = \cos z,$ etc.

The word 'sine'

Thinking of the sine constructed within a circle, Āryabhata called it *ardhā-jyā*, meaning 'half-chord', and then abrreviated it to *jyā* ('chord'). From *jyā*, the Arabs derive *jiba* which was then written *jb*. Later writers substituted *jaib*, a good Arbabian word meaning 'cove' or 'bay'. Still later, Gherado of Cremona (ca 1150) translated *jaib* into the Latin equivalent *sinus*, whence came our present *sine*.

Quiz 3.2

1. Function $\sin z$ is periodic, of period

2. $\frac{d}{dz}(\cos z) =$

3. $\sin z = 0 \Leftrightarrow z = n\pi \quad (n \in \mathbf{Z})$.
(a) True ✔ ; (b) False .

4. If $z = x + iy$ then $|\sin z|^2 = \sin^2 x + \sinh^2 y$.
(a) True ✔ ; (b) False .

5. If $z = x + iy$ then $|\sin z| \le |\sin x|$.
(a) True ✔ ; (b) False .

CHECK

Logarithmic Function

Does the exponential function have an inverse logarithmic function?

Since the exponential function is periodic, any inverse would have to be multi-valued. Let us write

$$w = \log z \Leftrightarrow z = \exp w.$$

If we set $z = r \operatorname{cis} \theta$, $w = u + iv$, then $r \operatorname{cis} \theta = e^u \operatorname{cis} v$.

From this, we deduce that

$$r = e^u, \quad u = \ln r, \quad v = \theta + 2k\pi.$$

That is,

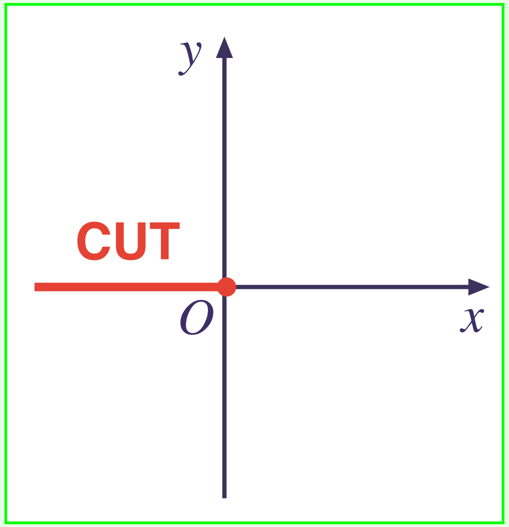
$$w = \log z = \ln |z| + i(\theta + 2k\pi) \ (k \in \mathbf{Z}).$$

Thus there are infinitely many values of $\log z$, the different values differing by $2k\pi i$. Each value of k gives a **branch** of the logarithm.

The Cut Plane

With $\log z = \ln|z| + i(\theta + 2k\pi)$ let us take $-\pi < \theta \le \pi$. Make a (red) cut in the complex plane along the negative x-axis. For any fixed value of k, we obtain a branch which does not cross this cut. So in the cut plane, each branch is single-valued. In particular we have the **principal branch**

$$\mathbf{Log}\, z = \ln r + i\theta \quad (-\pi < \theta \le \pi).$$



Notes

1. A path which crosses the cut moves to the next branch.

2. If z is real and positive, then $\operatorname{Log} z = \ln r$.

3. We can think of the branch planes interleaved together, with the x-axis as a common axis. A path drawn about the origin in one branch plane reaches the cut and then passes to the next branch plane.

4. Our choice of the positive x-axis for the cut was somewhat arbitrary. Other branch cuts are possible; but O is common to them all – O is a **branch point**.

Properties of the Logarithm (I)

Consider $\text{Log}\, z = \ln r + i\theta$ $(-\pi < \theta \leq \pi,\ r > 0)$ – that is, over the *open* domain excluding the cut. There are difficulties on the cut, for θ is not continuous there for any branch. Hence, for example, the Log function is not continuous on the cut, and so the Log function is not differentiable there.

1. $\text{Log}\, z$ is analytic over the open domain $(-\pi < \theta < \pi,\ r > 0)$.

Writing $\text{Log}\, z = u + iv$, we have $u = \frac{1}{2}\ln(x^2 + y^2)$, $v = \theta = \arctan {}^{y}/_{x}$. Hence

$$u_x = \frac{x}{x^2+y^2};\quad u_y = \frac{y}{x^2+y^2};\quad v_x = \frac{-y}{x^2+y^2};\quad v_y = \frac{x}{x^2+y^2}.$$

These functions are continuous on the given domain and satisfy the Cauchy-Riemann equations there. Hence by Theorem 2.5, $\text{Log}\, z$ is analytic.

[**Note** There is a problem in defining arctan here when $x = 0$. We could overcome this by defining $\theta = \text{arccot}\, {}^{x}/_{y}$, or by taking time to develop a polar form of the Cauchy-Riemann equations.]

Properties of the Logarithm (II)

2. Derivative

$$\frac{d}{dz}(\mathrm{Log}\, z) = \frac{1}{z} ,$$

$$\frac{d}{dz}(\mathrm{Log}\, z) = u_x + iv_x = \frac{x - iy}{x^2 + y^2} = \frac{1}{z} .$$

All branches have the same derivative, since they differ by a constant.

3. Inverse Property

$$\exp(\log z) = z \text{ (for any branch)}$$

$$\log(\exp z) = z \text{ (for a particular branch).}$$

4. Sums and Differences

$$\log z_1 + \log z_2 = \log(z_1 . z_2)$$

$$\log z_1 - \log z_2 = \log(z_1 / z_2)$$

providing we choose the appropriate logarithm branch on the right.

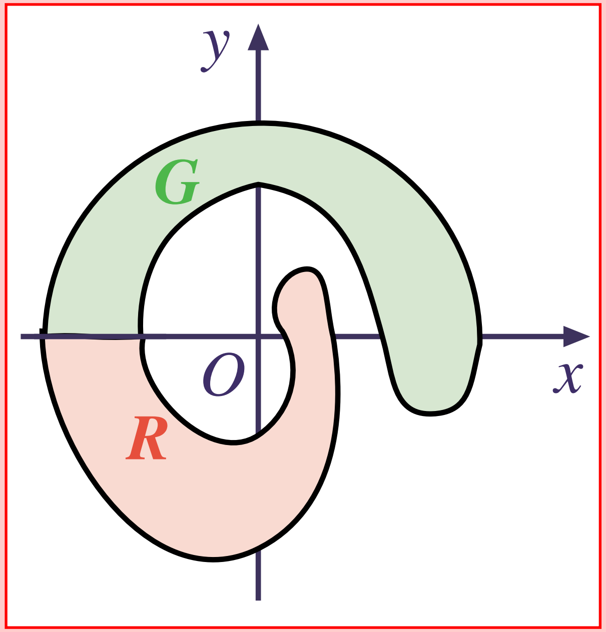
Examples on the Logarithm

Example 1. Evaluate $\text{Log}(-1) + \text{Log}(-1)$.

Now $-1 = 1 \,.\, \text{cis}\,\pi$, so $\text{Log}(-1) = 0 + i\pi$.

Hence $2\text{Log}(-1) = 2\pi i = \log 1$, but not $\text{Log}\,1\ (= 0)$.

Example 2. Show how to make $f(z) = \log z$ analytic on the open region $A = G \cup R$.



In the (green) region G, we define $f(z) = \text{Log}\,z$ (the principal value). In the (red) region R, we choose a different branch of the logarithm, defining

$$f(z) = \log|z| + i\arg z \quad (\pi < \arg z < 3\pi).$$

This definition allows a continuous transition acoss the cut.

Quiz 3.3

1. The function $\log z$ is
(a) single-valued ✔ ; (b) multiple-valued ☐ .

2. $\log(z_1 z_2) = \log z_1 + \log z_2$.
(a) True ✔ ; (b) False ☐ .

3. Give the value of $\text{Log}\, 1$.

☐

4. It is always true that $\text{Log}(z_1/z_2) = \text{Log}\, z_1 - \text{Log}\, z_2$.
(a) True ✔ ; (b) False ☐ .

5. For $\text{Log}\, z$, the range of the argument is:

☐

CHECK

Complex exponents

Using our knowledge of real powers, we define the complex power z^c $(c \in \boldsymbol{C})$ by

$$z^c = \exp(c \log z), \; (z \neq 0).$$

Since z^c is defined in terms of the logarithm, we expect z^c to be multivalued, so we use the cut plane as for the logarithm. Then since $\log z$ is single-valued and analytic in the cut plane, so is z^c.

Now

$$\frac{d}{dz}(z^c) = \frac{d}{dz}(\exp(c \log z)) = \exp(c \log z).\frac{z}{c} = z^c.\frac{z}{c} = cz^{c-1}.$$

So

$$\frac{d}{dz}(z^c) = cz^{c-1}.$$

Exponent examples

1. $i^{1/4} = \exp(\frac{1}{4}\log i) = \exp(\frac{1}{4}\, i\, (\frac{\pi}{2} \pm 2k\pi)) = \exp(\frac{\pi i}{8} \pm \frac{k\pi i}{2})$ – four values.

2. $i^{i} = \exp(i \log i) = \exp(i\,(\frac{\pi}{2} \pm 2k\pi)\, i) = \exp(-\frac{\pi}{2} \pm 2\mathrm{k}\pi)$.

The principal value is $\exp(-\frac{\pi}{2})$.

3. What is the relationship between $\exp z$ and e^{z}?

Clearly $e^{z} = \exp(z \log e)$.

Now $e = e \operatorname{cis} 0$, so $\log e = 1 \pm 2k\pi i$, and $\exp(z \log e) = \exp(z \pm 2k\pi i z)$.

It follows that $e^{z} = \exp z \,.\, \exp(2k\pi i z)$.

Setting $k = 0$ gives $e^{z} = \exp z$.

Thus $\exp z$ is the principal value of the multi-valued power function e^{z}.

Quiz 3.4

1. $\frac{d}{dz}(z^{i}) = iz^{i-1}$.

(a) True [✔] ; (b) False [ ] .

2. The principal value of i^{i} is

3. If $z \neq 0$ and k is real, then $|z^{k}| = |z|^{k}$.

(a) True [✔] ; (b) False [ ] .

4. $i^{1/3}$

(a) is single-valued [✔] ;
(b) has 3 values [ ] ;
(c) has infinitely many values [ ] .

5. z^{n}, $(n \in \boldsymbol{Z})$

(a) is single-valued [✔] ;
(b) has n values [ ] ;
(c) has infinitely many values [ ] .

CHECK

Theorem 2.5 Let $f = u + iv$ as before.

Suppose

(i) u, v, u_x, v_x, u_y, v_y exist in the neighbourhood of (x_0, y_0),

(ii) u_x, v_x, u_y, v_y are continuous at (x_0, y_0),

(iii) the Cauchy-Riemann equations are satisfied at (x_0, y_0).

RETURN

Example

The real functions $u = e^x \cos y$, $v = e^x \sin y$ are defined and continuous everywhere. So are u_x, v_x, u_y, v_y and you can easily check that the Cauchy-Riemann equations are satisfied.

Hence the function

$$f(z) = e^x \cos y + i e^x \sin y$$

is differentiable everywhere.

Since $u_x = u$, $v_x = v$,

$$f'(z) = f(z)$$

$$(= e^x \text{cis } y = e^{x+iy} = e^z).$$