4. MAPPING BY ELEMENTARY FUNCTIONS

There are many practical situations where we try to simplify a problem by transforming it. We investigate how various regions in the plane are transformed by elementary analytic functions.

You may find the following website helpful:

http://www.malilla.supereva.it/Pages/Fractals/Cmapper/cm.html

Linear Functions (I)

1. $w = f(z) = z + c \quad (c = (c_1, c_2) \in \mathbb{C})$ – this is a **translation** through c.
Each $z = (x, y)$ maps to $w = (x + c_1, y + c_2)$.

2. Let $b = |b| \operatorname{cis} \theta$ be a complex constant and let $w = f(z) = bz$.
If $z = r \operatorname{cis} \phi$, then $w = r.|b|.\operatorname{cis}(\theta + \phi)$.
That is, each point (r, θ) maps to $(|b|.r, \theta + \phi)$.
Thus we have a **rotation** about O through ϕ and an **enlargement** through $|b|$.

Linear Functions (II)

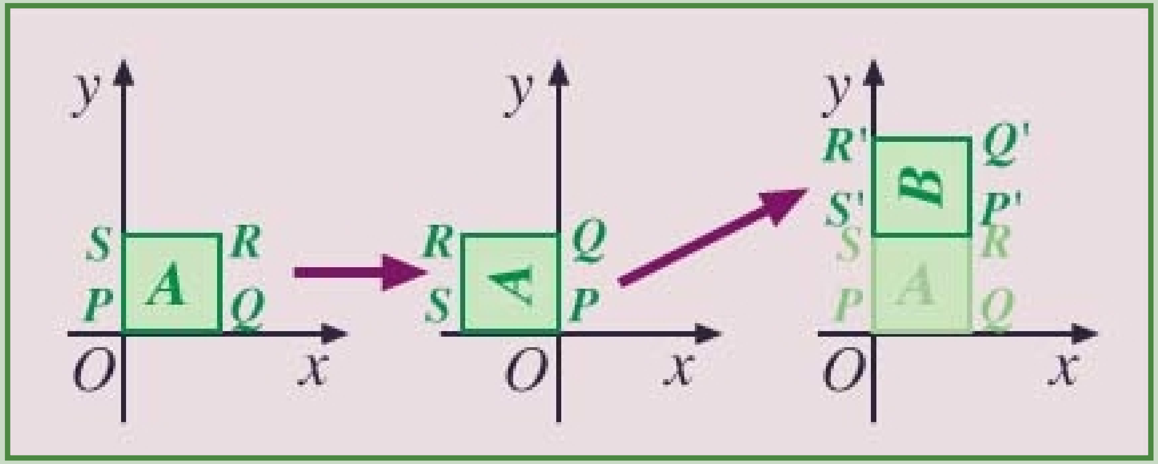
The general linear function is the mapping $w = f(z) = bz + c$. This is the rotation and enlargement (**2**), followed by the translation (**1**) previously discussed. That is,

$$z \rightarrow bz \rightarrow bz + c.$$

Example

Consider the transformation $w = iz + (i + 1)$.
This maps square A to square B in the figure below. Notice that S is an invariant point of this transformation. Geometrically, we have:

$$\text{rotation} \times \text{translation} = \text{rotation}.$$



The Function $f(z) = z^2$ (I)

Let $w = f(z) = z^2$. Then if $z = r \operatorname{cis} \theta$ and $w = \rho \operatorname{cis} \phi$, we have

$$\rho \operatorname{cis} \phi = r^2 \operatorname{cis} 2\theta \quad \text{and} \quad (r, \theta) \rightarrow (\rho, \phi) = (r^2, 2\theta).$$

Example

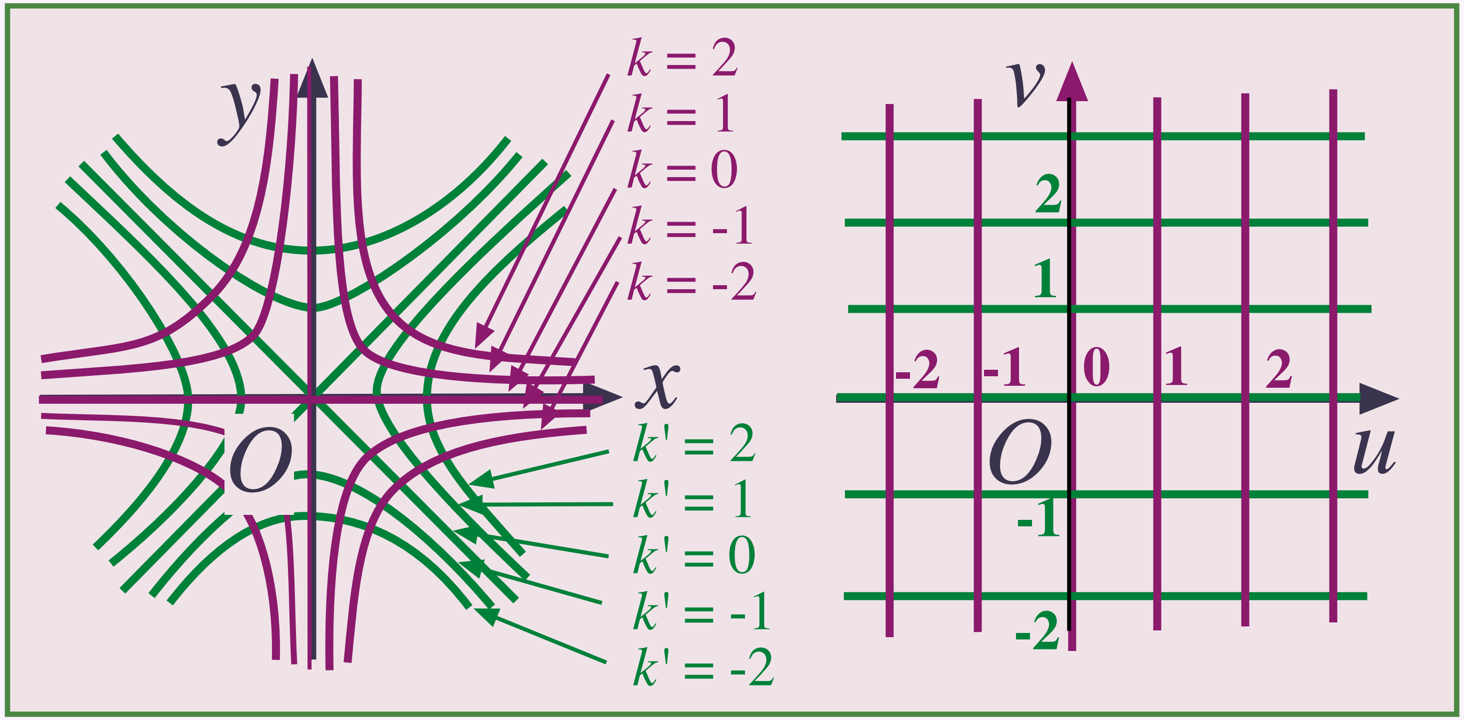
The first quadrant of the complex plane maps to the top half-plane under this mapping, since the angle range $0 \leq \theta \leq {}^{\pi}/_{2}$ maps to the range $0 \leq \phi \leq \pi$.

We note that in general, the mapping $w = f(z) = z^2$ will not be $1-1$.
For example, the points $\pm z$ both map to the same w.

The Function $f(z) = z^2$ (II)

Let us now write $w = f(z) = z^2$ in cartesian coordinates. Setting $w = u + iv$, $z = x + iy$, we obtain $u + iv = (x^2 - y^2) + 2xyi$. That is, $(x, y) \to (x^2 - y^2, 2xy)$.

So z-points on the hyperbola $x^2 - y^2 = k$ map to points on the straight line $u = k$. Similarly z-points on the hyperbola $2xy = k'$ map to points on the straight line $v = k'$.



Quiz 4.1

1. Under the mapping $w = iz + 2$ the point $1 + i$ is left invariant.

(a) True ☑ ; (b) False ☐ .

2. Under $w = iz + 2$, $(2, 0)$ is a vertex of the image of the square $0 \le x,\ y \le 1$.

(a) True ☑ ; (b) False ☐ .

3. The mapping $w = z^2$ with domain $\{z \mid z = r \operatorname{cis} \theta,\ {}^{-\pi}/_{4} < \theta < 0\}$ is $1-1$.

(a) True ☑ ; (b) False ☐ .

4. The mapping $w = z^2$ maps the second quadrant ${}^{\pi}/_{2} \le \theta \le \pi$ to a halfplane.

(a) True ☑ ; (b) False ☐ .

CHECK

The Function $f(z) = {}^1/_z$

The mapping $w = f(z) = {}^1/_z$ (equivalently $z = {}^1/_w$) sets up a 1–1 correspondence between points in the z- and w-planes excluding $z = 0, w = 0$.

In polar coordinates, $w = {}^1/_z$ becomes $\rho \operatorname{cis} \phi = \frac{1}{r} \operatorname{cis}(-\theta)$.

This transformation is the product of two simpler transformations:

$$z = r \operatorname{cis} \theta \;\rightarrow\; z' = \frac{1}{r} \operatorname{cis} \theta \quad \text{and} \quad z' = \frac{1}{r} \operatorname{cis} \theta \;\rightarrow\; w = \overline{z}'$$

– inversion in the unit circle $\odot$ followed by reflection in the x-axis.

Notes

1. Under inversion, the points on $\odot$ remain invariant.

2. Under $w = {}^1/_z$, (and under inversion),

the centre of $\odot$ remains invariant;
lines through O map to lines through O;
the interior of $\odot \leftrightarrow$ the exterior of $\odot$;
(in fact, all circles centred at O map to circles centred at O).

Mapping Circles and Lines under $f(z) = {}^1/_z$ (I)

Question What is the effect of $f(z) = {}^1/_z$ on more general lines and circles?

Expressing $w = f(z) = {}^1/_z$ in terms of cartesian coordinates, we obtain

$$w = u + iv = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

Thus

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2}$$

and inversely

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2}.$$

To
catch a lion in
the Sahara Desert,
place a spherical cage in
the desert, enter it, and
lock it. Now perform an
inversion with respect to the
cage. The lion is then in
the interior of the cage,
and we are outside.
H. Pétard

Mapping Circles and Lines under $f(z) = {}^1/_z$ (II)

Now consider the equation

$$a(x^2 + y^2) + bx + cy + d = 0 \quad (a, b, c, d \in \boldsymbol{R}).\ (*)$$

If $a \neq 0$, this is the equation of a circle; if $a = 0$, the equation of a straight line.

Substituting for x, y in terms of u, v we get

$$d(u^2 + v^2) + bu - cv + a = 0 \quad (\dagger)$$

Points (u, v) satisfying $(\dagger)$ correspond to points (x, y) satisfying $(*)$. Hence we have the four cases:

(a) $a \neq 0,\ d \neq 0$. Circle not through O $\rightarrow$ circle not through O.
(b) $a \neq 0,\ d = 0$. Circle through O $\rightarrow$ straight line not through O.
(c) $a = 0,\ d \neq 0$. Straight line not through O $\rightarrow$ circle through O.
(d) $a = 0,\ d = 0$. Straight line through O $\rightarrow$ straight line through O.

Example of a Mapping under $f(z) = {}^{1}/_{z}$

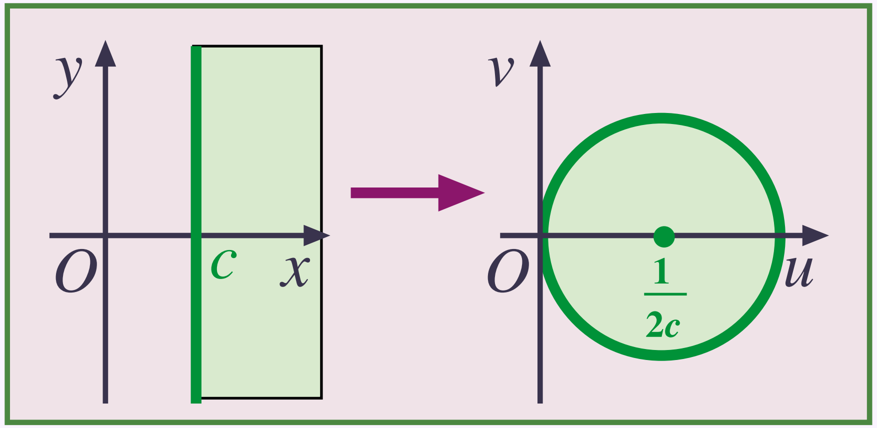
What is the image under $w = f(z) = {}^{1}/_{z}$ of the line $x = c$?

The line $x = c$ maps to the set of points satisfying

$$\frac{u}{u^2 + v^2} = c, \quad \text{that is,} \quad u^2 + v^2 - \frac{u}{c} = 0, \quad \text{or} \quad \left(u - \tfrac{1}{2}c\right)^2 + v^2 = \left(\tfrac{1}{2}c\right)^2.$$

This is the circle with centre $\left(\frac{1}{2c}, 0\right)$, passing through the origin O.

The half plane $x > c$ maps to the interior of the disc.



QUIZ 4.2

1. Under the mapping $w = {}^1/_z$ the image of $3 + 2i$ is

2. Under $w = {}^1/_z$, the circle $|z - 1| = 1$ maps to

3. Under $w = {}^1/_z$, the halfplane $y > 1$ maps to

4. Under $w = {}^1/_z$, any circle centred at O maps to itself.

(a) True ; (b) False .

CHECK

Bilinear Transformations

Let T be the transformation $w = \dfrac{az + b}{cz + d}$, $(ad - bc \neq 0,\ a, b, c, d \in \boldsymbol{C})$.

This is called a **bilinear transformation** as it can be rewritten as

$$cwz + dw - az - b = 0$$

– an equation which is linear in each of its variables, z and w.

Solving for w in terms of z, we see that the inverse of T is another bilinear transformation:

$$T^{-1} : \quad z = \frac{-dw + b}{cw - a} \ .$$

Notes

1. The singular points for T, T^{-1} are $z = {}^{-d}/_{c}$, $w = {}^{a}/_{c}$ respectively.

As each mapping has just one singular point, we write

$$z = {}^{-d}/_{c} \rightarrow \text{‘}w = -\infty\text{’}, \quad w = {}^{a}/_{c} \rightarrow \text{‘}z = -\infty\text{’}.$$

2. The set of bilinear transformations forms a **group**.

Understanding the Bilinear Transformation

In the formula for a bilinear transformation, if $c \neq 0$, we can write

$$w = \frac{az + b}{cz + d} = \frac{a(z + {}^{d}/_{c}) + (b - {}^{ad}/_{c})}{c(z + {}^{d}/_{c})} = \frac{a}{c} + \frac{(b - {}^{ad}/_{c})}{cz + d}$$

Setting $z' = cz + d$, $z'' = {}^{1}/_{z'}$ we obtain

$$w = \frac{a}{c} + \frac{(bc - ad)}{c} z'' . \qquad (*)$$

If $c = 0$, we get an expression of type (*) immediately.

It follows that any bilinear transformation can be obtained as the composition of linear transformations and the mapping $f(z) = {}^{1}/_{z}$, all of which map lines and circles to lines and circles.

Hence any bilinear transformation maps the set of lines and circles to itself.

Images under a Bilinear Transformation

The following reult will be useful.

Theorem 4.1 There exists a unique bilinear transformation which maps three given distinct points z_1, z_2, z_3 onto three distinct points w_1, w_2, w_3 respectively.

Proof The algebraic expression for the bilinear transformation can be written as

$$cwz + dw - az - b = 0.$$

This is an equation in four unknowns a, b, c, d. Hence the three ratios $a : b : c : d$ of these numbers are determined by substituting three pairs of corresponding values of z_i, w_i $(1 \leq i \leq 3)$.

In practice, this means that in general if we allocate image points to three points in the z-plane, then the bilinear transformation will be completely determined. Conversely, if we want to find the image of a circle (say) under a given bilinear transformation, then it is sufficient to find the images of three points on the circle.

Bilinear Transformation: Example

Let $a \in \boldsymbol{C}$ be constant with $\operatorname{Im} a > 0$. Find the image of the upper half plane ($y \geq 0$) under the bilinear transformation

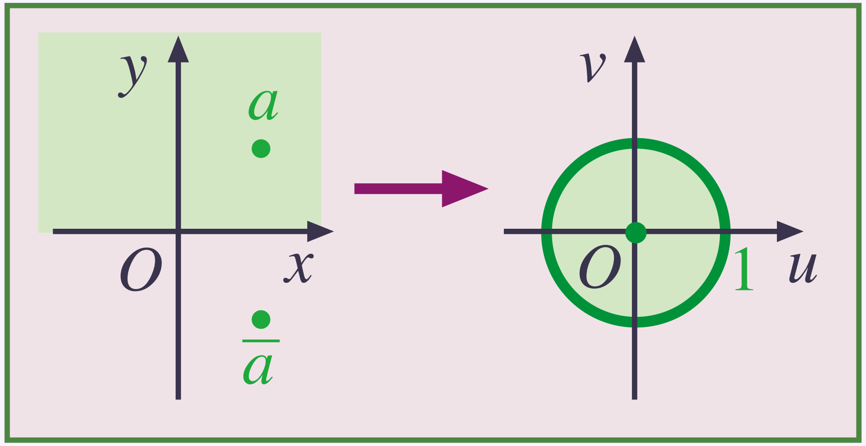
$$w = \frac{z - a}{z - \bar{a}}.$$

We first consider the boundary. For z on the x-axis, we have $|z - a| = |z - \bar{a}|$. Hence for such points,

$$|w| = \frac{|z - a|}{|z - \bar{a}|} = 1.$$

That is, the x-axis maps to the unit circle having centre O.

Also, $z = a$ (a point in the upper half plane) maps to $w = 0$ (a point interior to the unit circle). Observing that the mapping is continuous, we deduce that the image of the upper half plane is the interior of the disc.



QUIZ 4.3

1. The bilinear transformation which maps
$0 \to -1, -1 \to 0, 1 \to \infty$ maps $i \to$ ______

2. The image of the x-axis under $w = \dfrac{z-i}{z+1}$
is ______

3. The image of the y-axis under $w = \dfrac{z-i}{z+1}$
is ______

4. A bilinear transformation mapping a circle to a circle must map the interior of one to the interior of the other.
(a) True ☑ (b) False ☐ .

CHECK

The Transformation $w = \exp z$

As before, we set $z = x + iy$, $w = \rho \operatorname{cis} \phi$.
Then $w = \exp z$ gives $\rho = e^x$ and $\phi = y$.

The line $y = c$ maps onto the ray $\phi = c$ (excluding the end-point O) in a $1-1$ fashion. Similarly, the line $x = c$ maps onto the circle $\rho = e^c$. However, here, an infinite number of points on the line map to each image point.

Combining these results, we see that the rectangular region

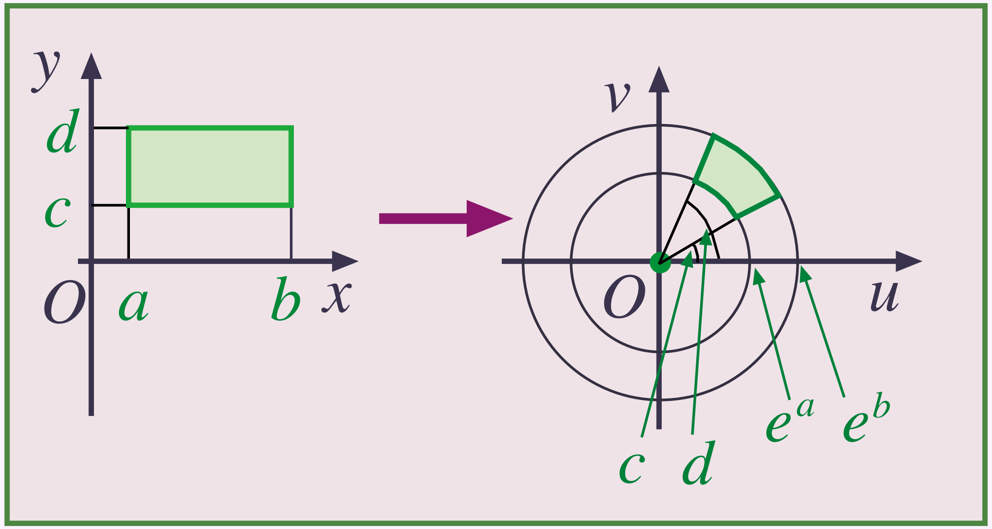
$$a \leq x \leq b, \quad c \leq y \leq d$$

is mapped to the region

$$e^a \leq \rho \leq e^b, \quad c \leq \phi \leq d,$$

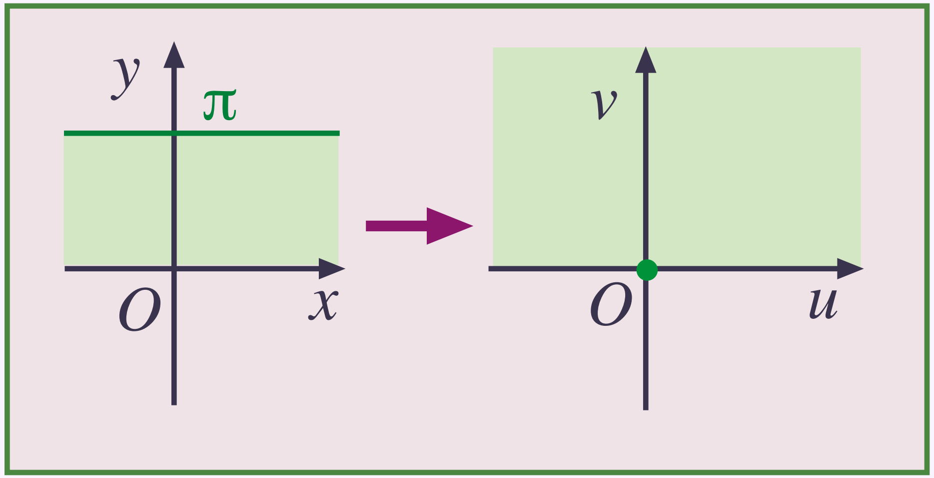
bounded by portions of circles and rays.

This mapping is $1-1$ if $d - c < 2\pi$.



Special Case of $w = \exp z$

As a particular case, under the exponential mapping, the strip $0 < y < \pi$ maps to the upper half plane $\rho > 0,\ 0 < \phi < \pi$ of the w-plane.



It is interesting to map the boundary points here.

This mapping is useful in applications.

QUIZ 4.4

1. Under the mapping $w = \exp z$,
the image of the x-axis is unbounded.

(a) True ✔ ; (b) False ☐ .

2. Under the mapping $w = \exp z$,
the image of the y-axis is unbounded.

(a) True ✔ ; (b) False ☐ .

3. Under $w = \exp z$, equally spaced lines
parallel to the y-axis map to
equally spaced concentric circles.

(a) True ✔ ; (b) False ☐ .

4. Under $w = \exp z$, the line $y = c$
maps to a ray in the w-plane. Then
the opposite ray arises from the line $y = -c$.

(a) True ✔ ; (b) False ☐ .

CHECK