5. COMPLEX INTEGRATION (A)

Definite integrals

Integrals are extremely important in the study of functions of a complex variable. The theory is elegant, and the proofs generally simple. The theory is put to much good use in applied mathematics.

We shall study *line integrals* of $f(z)$. In order to do this we shall need some preliminary definitions.

Let $F(t) = U(t) + iV(t)$ $(a \le t \le b)$ where U, V are real-valued, piecewise-continuous functions of t on $[a, b]$, i.e. continuous except for at most a finite number of jumps.

The **definite integral** of F on the interval $a \le t \le b$ is now defined by:

Definition $$\int_a^b F(t)\,dt = \int_a^b U(t)\,dt + i\int_a^b V(t)\,dt.$$

Properties of the definite integral

1. $\mathrm{Re}\left(\int_a^b F(t)\,dt\right) = \int_a^b U(t)\,dt = \int_a^b \mathrm{Re}\,(F(t))\,dt.$ **[Real part property]**

2. $\int_a^b kF = k\int_a^b F\,;\ (k \in \boldsymbol{C},\ \text{const}).$ **[Scalar multiple property]**

3. $\int_a^b (F+G) = \int_a^b F + \int_a^b G.$ **[Addition property]**

4. $\int_a^b F = -\int_b^a F.$ **[Interchange endpoints property]**

5. $\left|\int_a^b F\right| \le \int_a^b |F| \quad (a \le b)$ **[Modulus property]**

Property (1) follows immediately from the definition.

The proofs of (2), (3), (4) are trivial, and follow from the properties of real integrals.

We see from (2) and (3) that the integral behaves in a *linear* way.

Proof of Property (5)

5. $\left|\int_a^b F\right| \le \int_a^b |F| \quad (a \le b)$ **[Modulus property]**

Proof By definition, $\int_a^b F\, dt$ is a complex number.

So we can set $r_0 \operatorname{cis} \theta_0 = \int_a^b F\, dt$.

By Property (2), $r_0 = \int_a^b F \operatorname{cis}(-\theta_0)\, dt$.

Each side is real, and when a complex number is real, it is the same as its real part.

So by Property (1), $r_0 = \int_a^b \operatorname{Re}(F \operatorname{cis}(-\theta_0))\, dt$.

But $\operatorname{Re}(F \operatorname{cis}(-\theta_0)) \le |F \operatorname{cis}(-\theta_0)| = |F|.|\operatorname{cis}(-\theta_0)| = |F|$.

So $r_0 \le \int_a^b |F|\, dt$, providing $a \le b$.

Hence $\left|\int_a^b F\, dt\right| \le \int_a^b |F|\, dt$.

Arcs

Definition A **continuous arc** is a set of points (x, y): $x = \phi(t)$, $y = \psi(t)$ $(a \le t \le b)$, where ϕ, ψ are real continuous functions of the real parameter t.

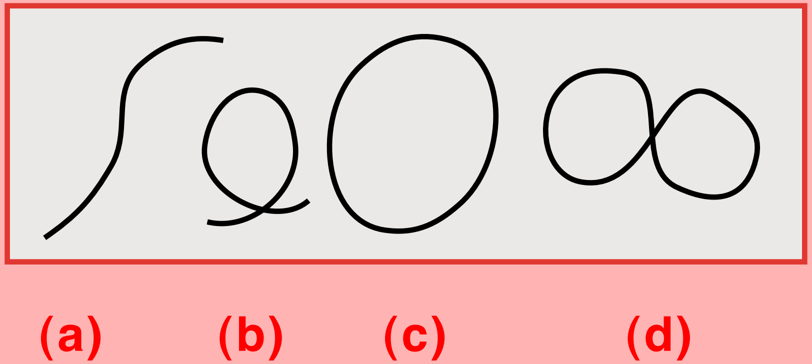
Notes

1. The definition gives a continuous mapping of $[a, b]$ to the arc with a corresponding ordering of points.
2. If no two distinct values of t map to the same point (x, y), the arc is a **simple arc** or a **Jordan arc**.
3. If Note (2) holds except that $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$, the arc is a **simple closed curve** or **Jordan curve**.

Examples

In the adjacent figure we have:

(a) Jordan arc
(b) Non-simple curve
(c) Jordan curve
(d) Non-simple closed curve.

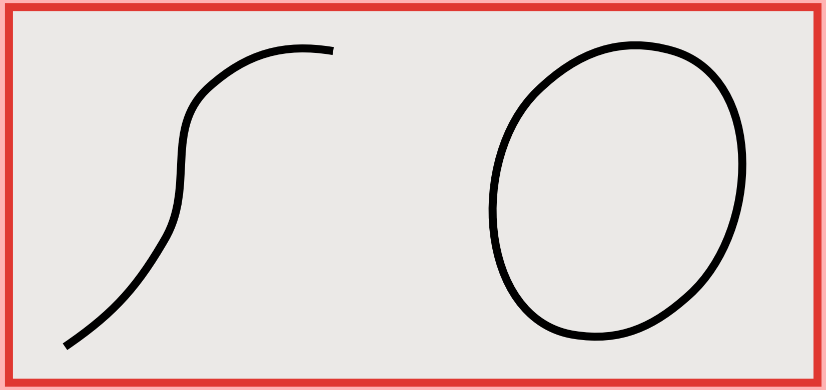


Contours

If functions ϕ, ψ have continuous derivatives $\phi'(t), \psi'(t)$ not simultaneously zero, we say the arc $(x, y) : x = \phi(t),\ y = \psi(t)\ (a \le t \le b)$ is **smooth** (has a continuously turning tangent).

Definition A **contour** is a continuous chain of a finite number of smooth arcs joined end to end.

Examples



Contour **Simple closed contour**

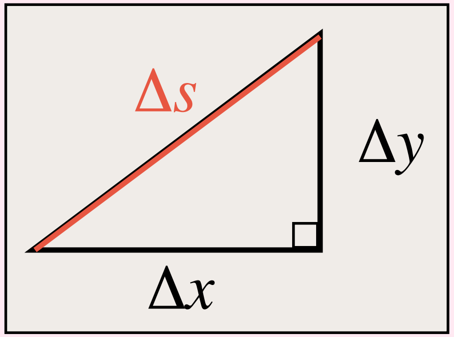
Length of an arc

Definition For a smooth arc, the **length** exists and is given by

$$L = \int_a^b \sqrt{[\phi(t)]^2 + [\psi(t)]^2}\, dt,$$

providing $a \le b$.

Now, where does this strange formula come from?
From Pythagoras' Theorem, the length Δs of a small portion of the arc is given by



$$\Delta s = \sqrt{[\Delta x]^2 + [\Delta y]^2}\,.$$

Dividing through by Δt gives

$$\frac{\Delta s}{\Delta t} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}.$$

Integrating this expression with respect to t gives the required result.

Note It can be shown that this formula is independent of the choice of parameter used.

QUIZ 5.1A

1. The function $\phi(F) = \int_a^b F$ is a linear function.

(a) True ✔ ; (b) False ☐ .

2. If $a \le b$, then $\int_a^b |F| \le |\int_a^b F|$.

(a) True ✔ ; (b) False ☐ .

3. A smooth continuous arc is a contour.

(a) True ✔ ; (b) False ☐ .

4. If C is a contour, then C must be a smooth continuous arc.

(a) True ✔ ; (b) False ☐ .

CHECK

QUIZ 5.1B

We show $\left|\int_a^b F\right| \le \int_a^b |F| \quad (a \le b)$.

By definition, $\int_a^b F\,dt$ is a complex number. So we can set **{ 1. }** $= \int_a^b F\,dt$.

By Property (2), $r_0 =$ **{ 2. }**. Each side is real, and when a complex number is real, it is the same as its real part. So by Property (1), $r_0 =$ **{ 3. }**.

But $\mathrm{Re}(F \operatorname{cis}(-\theta_0)) \le$ **{ 4. }** $= |F|.|\operatorname{cis}(-\theta_0)| = |F|$.

So $r_0 \le \int_a^b |F|\,dt$, providing $a \le b$. Hence $\left|\int_a^b F\,dt\right| \le \int_a^b |F|\,dt$.

Match the above boxes 1, 2, 3, 4 with the selections

(a) $\int_a^b F \operatorname{cis}(-\theta_0)\,dt$, (b) $|F \operatorname{cis}(-\theta_0)|$,
(c) $r_0 \operatorname{cis}\theta_0$, (d) $\int_a^b \mathrm{Re}(F \operatorname{cis}(-\theta_0))\,dt$.

My solutions:

1. ____ **2.** ____ **3.** ____ **4.** ____

Line Integrals

Let C be a contour extending from $z = \alpha$ to $z = \beta$, and set $z = x + iy$.

Thus for z on C, $x = \phi(t)$, $y = \psi(t)$ $(a \le t \le b)$ say, where ϕ, ψ are continuous and ϕ', ψ' are sectionally continuous.

Also $t = a$ when $z = \alpha$, and $t = b$ when $z = \beta$.

Let $f(t) = u(t) + iv(t)$ be a (sectionally) continuous function on C (that is, the real functions $u = u(t)$ and $v = v(t)$ are sectionally continuous over $a \le t \le b$.)

Definition We define

$$\int_C f(z)\, dz = \int_a^b f\,[\phi(t) + i\,\psi(t)].[\phi'(t) + i\,\psi'(t)]\, dt.$$

Here, $\int_C f(z)\, dz$ is a **line integral**, or **contour integral**.

Note that the line integral exists because the integrand on right is sectionally continuous.

Expanding the Line Integral

Suppose $f(z) = u + iv = u(\phi(t), \psi(t)) + iv(\phi(t), \psi(t))$.

Then substituting $f(z) = u + iv$ in the defining expression

$$\int_c f(z)\, dz = \int_a^b f\,[\phi(t) + i\,\psi(t)].[\phi'(t) + i\,\psi'(t)]\,dt,$$

gives

$$\int_c f(z)\, dz = \int_a^b (u\phi' - v\psi')\,dt + i\int_a^b (u\psi' + v\phi')\,dt,$$

or simply

$$\int_c f(z)\, dz = \int_c (u\, dx - v\, dy) + i\int_c (u\, dy + v\, dx). \qquad (*)$$

Note In summary, $\int_c f(z)\, dz = \int_c (u + i\mathrm{v})(dx + i\, dy).$

Thus by (*) we have expressed the complex line integral in terms of two real line integrals.

Properties of the Line Integral

1. $\int_{\beta}^{\alpha} f(z)\,dz = -\int_{\alpha}^{\beta} f(z)\,dz$ (taken over the same contour).

2. $\int_{\alpha}^{\beta} kf(z)\,dz = k\int_{\alpha}^{\beta} f(z)\,dz$ ($k \in \boldsymbol{C}$, constant).

3. $\int_{\alpha}^{\beta} [f(z) + g(z)]\,dz = \int_{\alpha}^{\beta} f(z)\,dz + \int_{\alpha}^{\beta} g(z)\,dz$.

4. If C_1 is a contour (with parameter *t* from) α to β, C_2 a contour from β to γ, and $C = C_1 \cup C_2$ a contour from α to γ, then $\int_{C} f(z) = \int_{C_1} f(z) + \int_{C_2} f(z)$.

5. If C has length L and $|f(z)| \le M$ on C, then $|\int_{C} f(z)\,dz| \le ML$.

Properties (2) and (3) express the *linearity* of the integral.

Properties (1) to (4) follow easily from known properties of the real integral.

Proof of Property 5

Proof We use Property 5 of the Definite Integral:

$$\left|\int_a^b F\right| \leq \int_a^b |F| \quad (a \leq b) \qquad (*)$$

Now,

$$\left|\int_c f(z)\, dz\right| = \left|\int_a^b f\,[\phi(t) + i\,\psi(t)].[\phi'(t) + i\,\psi'(t)]\,dt\right| \qquad (a \leq b)$$

$$\leq \int_a^b |f\,[\phi(t) + i\,\psi(t)].[\phi'(t) + i\,\psi'(t)]|\, dt \qquad \text{(by inequality (*))}$$

$$\leq M \int_a^b |\phi'(t) + i\,\psi'(t)|\, dt \quad (\text{ since } |f(z)| \leq M\,)$$

$$= M \int_a^b \sqrt{(\phi'(t))^2 + (\psi'(t))^2}\, dt$$

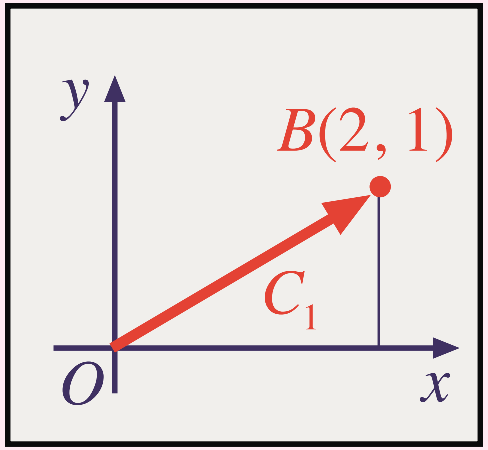
$$= ML\,.$$

Line Integral Example I

Find $I_1 = \int_{C_1} z^2\, dz$, where C_1 is the illustrated path OB.

Line OB has equation $x = 2y$.

Taking y as parameter we obtain the set of points $(2y, y)$ $(0 \le y \le 1)$.



Now, z^2 is continuous and on C_1,

$z^2 = (x^2 - y^2) + 2ixy = 3y^2 + 4y^2 i$,

$dz = dx + i\, dy = (2 + i)\, dy$ [strictly $(\phi'(y) + i\psi'(y))\, dy$].

Hence

$$I_1 = \int_0^1 (3y^2 + 4y^2 i)(2 + i)\, dy = (3 + 4i)(2 + i) \int_0^1 y^2\, dy = \tfrac{1}{3}\,.(2 + 11i).$$

Line Integral Example II

Find $I_2 = \int_{C_2} z^2\, dz$, where C_2 is the illustrated path OAB.

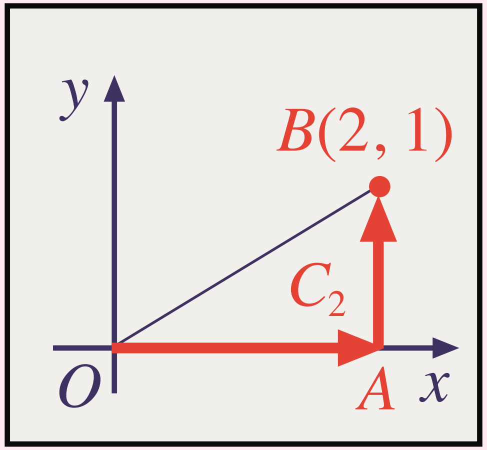
Now

$$I_2 = \int_{OA} z^2\, dz + \int_{AB} z^2\, dz$$

$$= \int_0^2 x^2\, dx + \int_0^1 [(4 - y^2) + 4iy]\; i\; dy$$

$$= \frac{8}{3} + [4 - \frac{1}{3} + 2i]$$

$$= \frac{1}{3}(2 + 11i).$$



Points on AB are $(2,\ 2 + iy)$, with $0 \le y \le 1$.

Some interesting questions

Question 1 We see that in the previous examples, the integral from O to B is the same for both contours. Is this a coincidence? Or does it always happen? If it doesn't always happen, when does it?

We can express this in a different way. We see that

$$\int_{OABO} = \int_{OAB} + \int_{BO} = I_2 - I_1 = 0.$$

So we ask: Is the integral around a closed contour always zero?

Question 2 We observe that if we forget the contour altogether, and simply integrate, then we obtain:

$$\frac{z^3}{3}\Big|_0^{2+i} = \tfrac{1}{3}(2+i)^3 = \tfrac{1}{3}(2+11i)$$

again!

Does this always happen? This would say that $\int_C f$ is independent of the contour C.

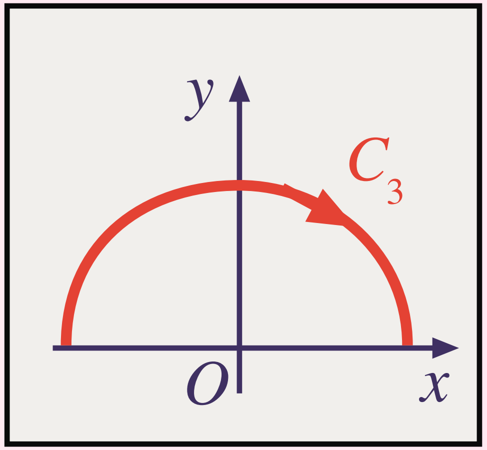
Line Integral Example III

Find $I_3 = \int_{C_3} \bar{z}\, dz$, where C_3 is the illustrated red path.

Now C_3 is the contour of points

$$\{ (x, y) \mid x = \cos\theta,\ \ y = \sin\theta,\ \ \pi \geq \theta \geq 0 \}$$

and $$dz = (-\sin\theta + i\cos\theta)\, d\theta.$$



Hence we have

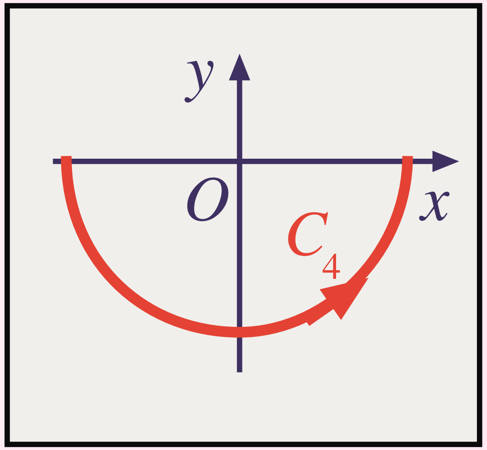
$$\begin{aligned} I_3 &= \int_{C_3} \bar{z}\, dz \\ &= \int_{\pi}^{0} (\cos\theta - i\sin\theta)(-\sin\theta + i\cos\theta)\, d\theta \\ &= \int_{\pi}^{0} i\, d\theta \\ &= -\pi i. \end{aligned}$$

Line Integral Example IV

Find $I_4 = \int_{C_4} \bar{z}\, dz$, where C_4 is the illustrated red path.

In this case we obtain

$$I_4 = \int_{C_4} \bar{z}\, dz$$
$$= \int_{\pi}^{2\pi} i\, d\theta$$
$$= +\pi i.$$



We observe that this is different from I_3!

And if we set $C = (-C_3) \cup C_4$, then $\int_C \bar{z}\, dz = 2\pi i$. (not 0) where the contour C is traversed in an anti-clockwise direction.

Note On the contour C, $|z|^2 = z\bar{z} = 1$. That is, $\bar{z} = {}^1/_z$. This suggests that there might be some special significance about the singular point $z = 0$ being inside the contour.

Line Integral Example V

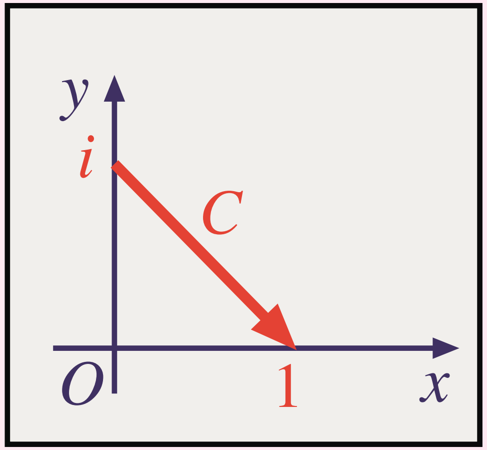
Without evaluating the integral, find an upper bound for $\left|\int_C {}^{dz}/_{z^4}\right|$ on the given contour C.

We make use of Property 5:

$$\left|\int_C f(z)\,dz\right| \le ML.$$

In this case, the length $L = \sqrt{2}$.

Also, the contour C has equation $y = 1 - x \quad (0 \le x \le 1)$.



Therefore for z on C

$$|z^4| = (x^2 + y^2)^2 = [x^2 + (1-x)^2]^2 = [2x^2 - 2x + 1]^2.$$

That is, $\quad |z^4| = [2(x - \tfrac{1}{2})^2 + \tfrac{1}{2}]^2 \ge \tfrac{1}{4}.$

(In retrospect, this is obvious from the figure! Why?)

Hence $\quad \left| {}^{1}/_{z^4} \right| \le 4 \quad \text{and} \quad |I| \le 4\sqrt{2}$.

QUIZ 5.2

1. If C is the line segment from O to $1+i$, then $\int_c 2z\ dz = \boxed{}$.

2. If C is the line segment from O to 1, followed by the line segment from 1 to $1+i$, then

$$\int_c 2z\ dz = \boxed{}.$$

3. If C is the upper half, from π to 0, of the circle $|z| = \sqrt{2}$, then

$$\int_c \bar{z}\ dz = \boxed{}.$$

4. If C is the circle centre O, radius k and fully traversed, then $\left|\int_c {}^{1}/_{z}\ dz\right| \leq 2\pi$.

(a) True ✔ (b) False ☐ .

CHECK

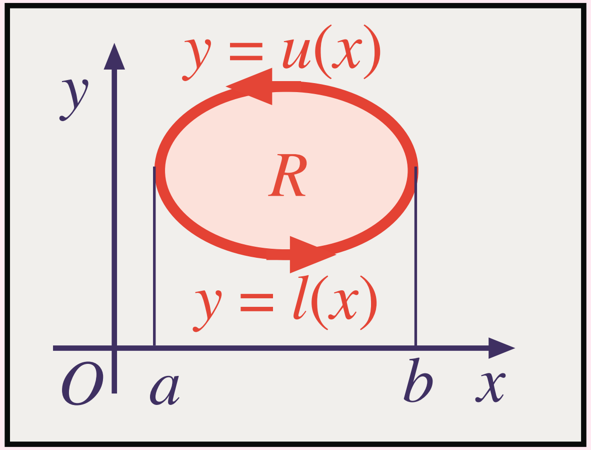
Green's Theorem

Let R be a closed region in the real plane made up of a closed contour C and all interior points.

If $P(x, y)$, $Q(x, y)$ are real continuous functions over R and have continuous first order partial derivatives, then Green's Theorem says:

$$\int_c (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$$

where C is described in the anti-clockwise direction.



Outline Proof

$$\iint_R P_y\,dx\,dy = \int_a^b \left[\int_{l(x)}^{u(x)} P_y\,dy\right] dx$$

$$= \int_a^b \left[P(x, u(x)) - P(x, l(x))\right] dx$$

$$= -\int_c P\,dx \qquad \text{etc.}$$

Cauchy's Theorem

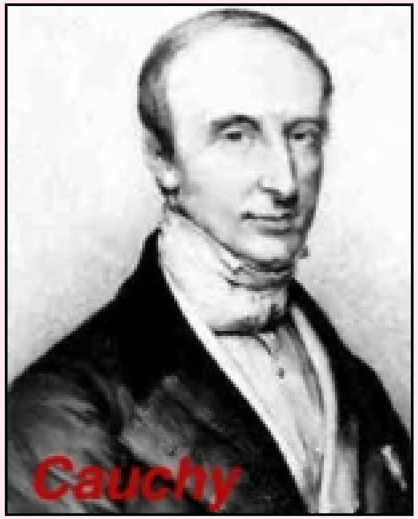
Now consider a function $f(z) = u(x, y) + iv(x, y)$ which is analytic at all points within and on the closed contour C, and is such that $f'(z)$ is continuous there.
We show that $\int_c f(z)\, dz = 0.$

The given conditions tell us that u, v and their first order partial derivatives are continuous. Now

$$
\begin{aligned}
\int_c f(z)\ dz &= \int_c (u\ dx - v\ dy) + i \int_c (u\ dy + v\ dx) \\
&= -\iint_R (v_x + u_y)\, dx\, dy + i \iint_R (u_x - v_y)\, dx\, dy \\
&= 0,
\end{aligned}
$$

using Green's Theorem and the Cauchy-Riemann equations.

This result was discovered by Cauchy.



Examples and the Cauchy-Goursat Theorem

As a consequence of Cauchy's Theorem we have the following:

Examples $\int_C dz = 0, \; \int_C z\, dz = 0, \; \int_C z^2\, dz = 0,$

for all closed contours C, since these functions are analytic everywhere and their derivatives are everywhere continuous.

Goursat showed that Cauchy's condition '$f'(z)$ is continuous' can be omitted. This discovery is important, because from it we can deduce that all derivatives of analytic functions are also analytic. But, proving this stronger result takes much more effort!

Theorem 5.1 (Cauchy-Goursat Theorem) If a function f is analytic at all points interior to and on a closed contour C, then

$$\int_C f(z)\, dz = 0.$$

Cauchy - Goursat Theorem, Example I

Find $\int_c \frac{z^2}{z-3}\,dz$ where the contour is the unit circle $|z| = 1$.

Now the integrand $z^2/(z-3)$ is analytic everywhere except at the point $z = 3$.

This point lies outside the circular disk $|z| \leq 1$.

Hence by the Cauchy-Goursat Theorem

$$\int_c \frac{z^2}{z-3}\,dz = 0.$$

Most of the useful applications of the Cauchy-Goursat Theorem are as yet beyond us, but the next example gives us a glimpse of what can be done.

Cauchy-Goursat Theorem, Example II

Assuming **Laplace's Integral**:

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

show that

$$\int_0^{\infty} e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

[We note that Laplace's integral can be evaluated by writing

$$\int e^{-x^2} dx \int e^{-y^2} dy = \iint e^{-x^2-y^2} dx \, dy$$

and expressing this second integral in polar coordinates.]

We integrate $f(z) = \exp(-z^2)$ around the rectangle C defined by $|x| \le R$, $0 \le y \le b$. Later we let $R \to \infty$.

Since $\exp(-z^2)$ is entire, the Cauchy-Goursat Theorem applies, that is, $\int_C e^{-z^2} dz = 0$.

[Continued]

Example II, continued

Hence

$$\int_{-R}^{R} e^{-x^2}\,dx + \int_0^b e^{-R^2+y^2-2iRyi}\,dy - \int_{-R}^{R} e^{-x^2+b^2-2xbi}\,dx - \int_0^b e^{-R^2+y^2+2iRyi}\,dy = 0.$$

Taking out the factor e^{-R^2} from the second and fourth integrals and letting $R \to \infty$, we get a zero contribution (using the $|\int dy| \leq ML$ formula). Hence letting $R \to \infty$,

$$\int_{-\infty}^{\infty} e^{-x^2+b^2-2xbi}\,dx = \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi} \ \text{(given)}.$$

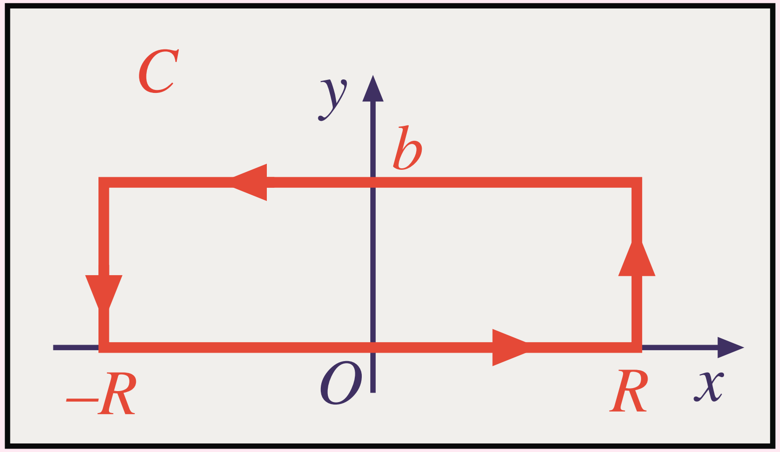
Equating the real parts,

$$e^{b^2}\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\,dx = \sqrt{\pi}$$

or

$$\int_0^{\infty} e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

since the integrand is even.



Indefinite Integrals

Let f be analytic in D and $z_0, z \in D$. Then for contours C_1, C_2 lying in D, joining points z_0, z, and being traversed from z_0 to z,

$$\int_{C_1} f - \int_{C_2} f = 0, \text{ by the Cauchy-Goursat Theorem.}$$

Theorem 5.2 ('Primitive' Theorem) For all such paths in the domain D

$$F(z) = \int_{z_0}^{z} f(z')\,dz'$$

has the same value, and $F'(z) = f(z)$.

Note This will allow us to evaluate some line integrals by straight integration. We shall see that the name of the theorem comes from the fact that the function F satisfying $F'(z) = f(z)$ is called a **primitive** of f.

Proof of the 'Primitive' Theorem

Proof Let $z + \Delta z$ be a point in D. Then

$$F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(z')\,dz' - \int_{z_0}^{z} f(z')\,dz' = \int_{z}^{z+\Delta z} f(z')\,dz',$$

where since Δz is very small, we may take the path from z to $z + \Delta z$ to be a line segment.

Now clearly on the segment, $\int_{z}^{z+\Delta z} dz' = \Delta z$. So

$$f(z) = \frac{f(z)}{\Delta z} \int_{z}^{z+\Delta z} dz' = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\, dz'.$$

Then

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(z') - f(z)]\, dz'.$$

[Continued]

Proof of the 'Primitive' Theorem (Continued)

Now f is continuous at z. Hence for all $\varepsilon > 0$, there exists $\delta > 0$:

$$|\Delta z| < \delta \quad \Rightarrow \quad |f(z') - f(z)| < \varepsilon .$$

Hence, when $|\Delta z| < \delta$,

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| \leq \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(z') - f(z)] \, dz' \right|$$

$$\leq \frac{1}{|\Delta z|} . \varepsilon . |\Delta z| = \varepsilon ,$$

(using Property 5).

That is,

$$\lim_{z \to z_0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z)$$

and $F(z)$ exists at each point of D, and $F'(z) = f(z)$.

Notes on the 'Primitive' Theorem

We say that F is an **indefinite integral** or **primitive** (**anti-derivative**) of f and write

$$F(z) = \int f(z)\,dz\,.$$

That is, F is an analytic function whose derivative is $f(z)$.

Since

$$\begin{aligned}\int_{\alpha}^{\beta} f(z)\,dz &= \int_{z_0}^{\beta} f(z)\,dz - \int_{z_0}^{\alpha} f(z)\,dz \\ &= F(\beta) - F(\alpha),\end{aligned}$$

we can use this as a means of evaluating line integrals.

(All paths here are assumed to be in the domain D.)

QUIZ 5.3

1. If $f(z) = {}^{z}/_{(z^2 - 4)}$, and C is the circle $|z - i| = 1$ traversed positively,

then $\int_c f(z)\,dz =$ ______

2. If in Q1 $f(z) = \mathrm{Log}(z+1)$,

then $\int_c f(z)\,dz =$ ______

3. If in Q1 $f(z) = {}^{1}/_{(z - i)}$,

then $\int_c f(z)\,dz =$ ______

4. If C and f are as in Q1, then the integrals from O to $2i$ along either semicircle of C are equal.

(a) True ✔ ; (b) False .

CHECK