5. COMPLEX INTEGRATION (B)

The 'Primitive Theorem' : Example 1

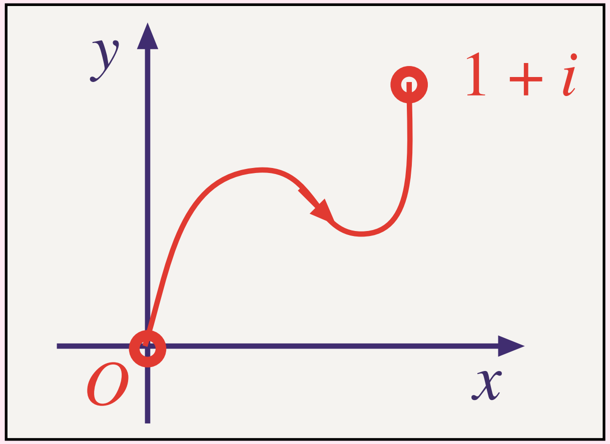
The function $f(z) = z^2$ is entire.

It has primitive $F(z) = \frac{1}{3}.z^3$.

So by our theorem,

$$\int_0^{i+1} z^2\,dz = \tfrac{1}{3}.z^3\Big]_0^{i+1}$$

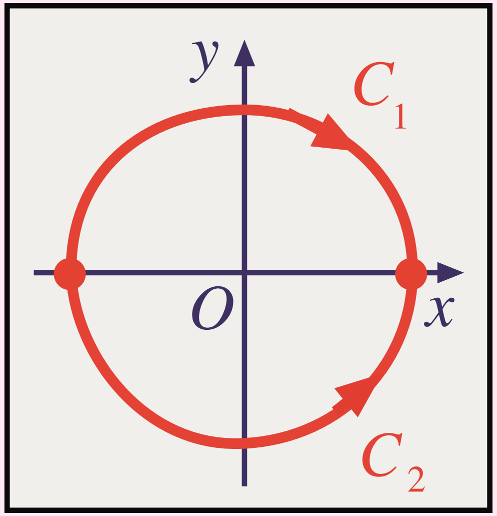
$$= \tfrac{1}{3}(1+i)^3,$$



along any contour joining 0 and $1 + i$.

The 'Primitive' Theorem: Example II

We recall that $\int_{C_1} {}^1/_z \, dz = -\pi i$, $\int_{C_2} {}^1/_z \, dz = \pi i$, where C_1, C_2 are upper and lower semicircles of the unit circle. Now we can easily choose our domain D to avoid O and contain the given contour C_i. **Mysterious question!** So why does the difference occur?



The answer is that for the different contours, we need different primitives!

For C_1, we might choose as primitive

$$\log z = \ln|z| + i \arg z \qquad ({}^{-\pi}/_2 < \arg z < {}^{3\pi}/_2).$$

Then

$$\int_{C_1} {}^1/_z \, dz = \log z \,\Big]_{-1}^{1} = -\pi i.$$

But for C_2, our choice of primitive could be

$$\log z = \ln|z| + i \arg z \qquad ({}^{\pi}/_2 < \arg z < {}^{5\pi}/_2)$$

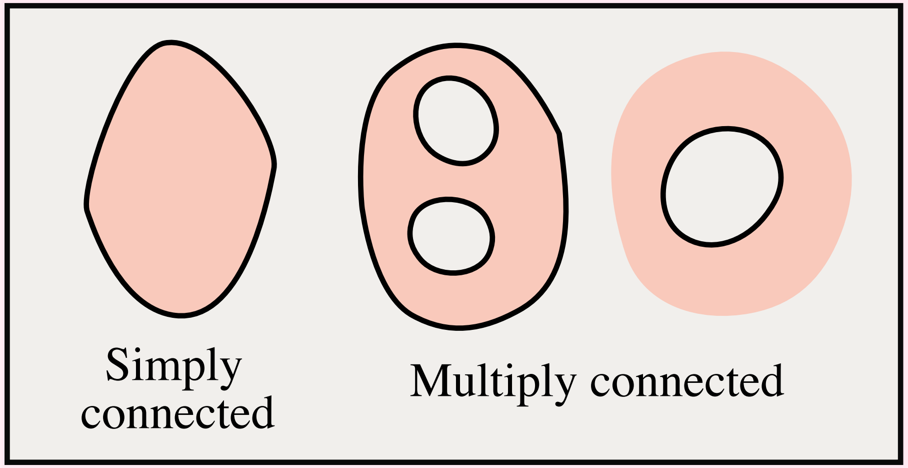
and

$$\int_{C_1} {}^1/_z \, dz = \log z \,\Big]_{-1}^{1} = 2\pi i - \pi i = \pi i.$$

Simply and Multiply Connected Domains

Defintion A **simply connected** domain D is an open connected region such that every closed contour within it encloses only points of D. Otherwise the domain is **multiply connected**.

Examples



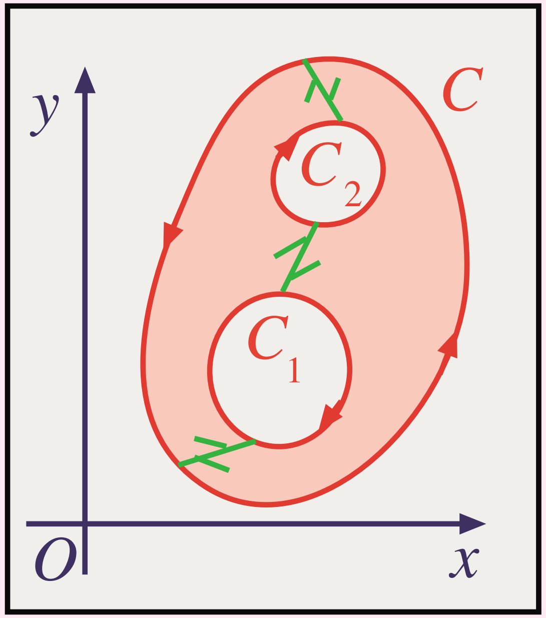
The Cauchy-Goursat Theorem has been stated for *simply* connected domains.

That is, if $f(z)$ is analytic throughout a simply connected domain D, then for every closed contour C within D, $\int_c f(z) = 0$.

Question IS the theorem still true for *multiply* conenected domains?

Extending the Cauchy-Goursat Theorem

It will be useful to extend the Cauchy-Goursat Theorem to certain multiply-connected domains. Consider the illustrated (red) region D and suppose that $f(z)$ is analytic over this (closed) region.



We assert that $\int_B f(z)\ dz = 0$, where B is the total directed boundary $(C \cup C_1 \cup C_2)$, with all components traversed so that the region is on the left.

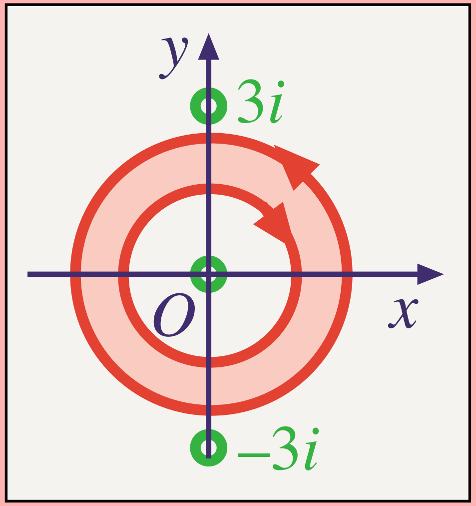
This is easy to prove.

We insert the indicated green links partitioning D into two simply-connected domains. We apply the Cauchy-Goursat Theorem to the boundaries of the left and right regions, obtaining two line integrals having value 0. Putting the two circuits together, the integrals along the introduced links cancel, giving the required result.

Multiply-Connected Domains: Examples

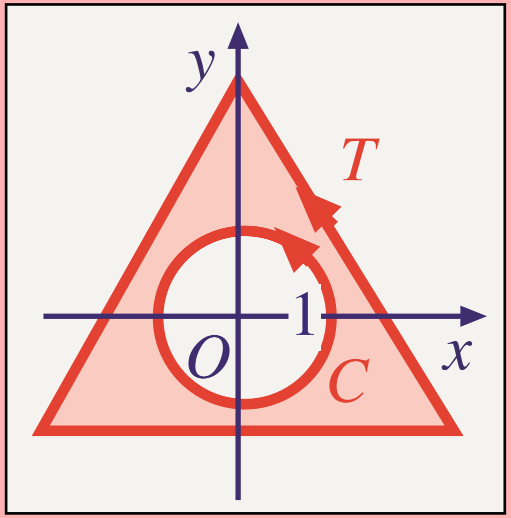
Example 1 $\int_B \frac{dz}{z^2(z^2+9)} = 0$

where B is the two-circle contour shown.



For, the integrand has singularities $0, -3i$, and is analytic over the enclosed domain.

Example 2 Find $\int_T \frac{dz}{z}$ where T is the illustrated triangle.



Now $1/z$ is analytic inside T, except at O. Consider the unit circle C, centre O. Since $1/z$ is analytic in the domain bounded by C and T, noting the direction of C,

$$\int_C - \int_T = 0, \text{ that is, } \int_C = \int_T .$$

Hence it is sufficient to evaluate the integral around C. So although we have not solved this problem, we have simplified it.

The Cauchy Integral Formula

Theorem 5.3 Let f be analytic everywhere within and on a closed contour C. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$

where the integral is taken in the positive sense around C.

Notes

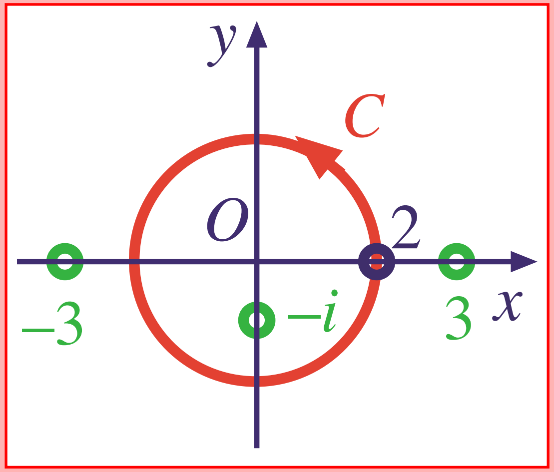
(1) The formula is the **Cauchy integral formula**. It is remarkable because it gives the value of f at z_0 in terms of the values of f on the boundary. That is, for an analytic function, fixing the boundary values completely determines f at points inside C.

(2) The theorem can also be used to evaluate certain integrals.

Cauchy Integral Formula Example

Evaluate

$$\int_c \frac{z}{(9 - z^2)(z + i)}\, dz$$



where C is $|z| = 2$, transversed once in the anticlockwise direction.

Take $f(z) = z/(9 - z^2)$, $z_0 = -i$. Then we observe that $f(z)$ is analytic in and on C. So using the Cauchy integral formula,

$$I = f(z_0) = \int_c \frac{f(z)}{z - (-i)}\, dz$$

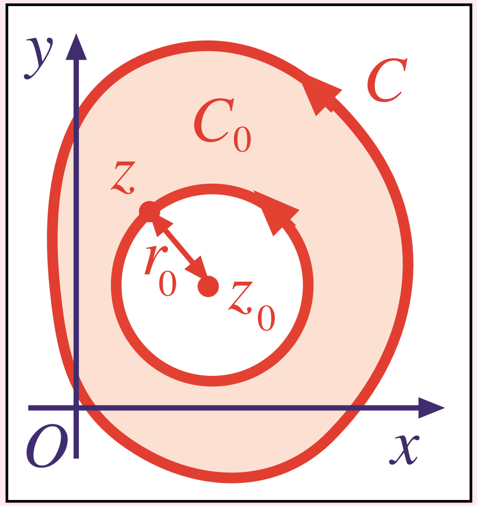
$$= 2\pi i.\ f(z_0)$$

$$= 2\pi i.\ \frac{-i}{10}$$

$$= \frac{\pi}{5}\ .$$

Proof of the Cauchy Integral Formula (I)

Let C_0 be a circle centre z_0, radius r_0 interior to C. Now the function $f(z)/(z-z_0)$ is analytic at all points in and on C except at $z = z_0$, in particular in the region D lying between C and C_0. Hence as in Example 2 above



$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_0} \frac{f(z)}{z - z_0}\, dz$$

where both contours are traversed in the positive direction.

Hence

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_{C_0} \frac{f(z_0) + f(z) - f(z_0)}{z - z_0}\, dz$$

$$= f(z_0) \int_{C_0} \frac{dz}{z - z_0} + \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz$$

or $\qquad I = f(z_0)\, I_1 + I_2$ say.

[Continued]

Proof of the Cauchy Integral Formula (II)

For z on C_0, we can write $z - z_0 = r_0 \operatorname{cis}\theta = r_0 \exp(i\theta)$.
Hence $dz = i r_0 \exp(i\theta)\, d\theta$, and

$$I_1 = \int_{C_0} \frac{dz}{z - z_0} = i \int_0^{2\pi} d\theta = 2\pi i \quad \text{for every } r_0 > 0.$$

Also, f is continuous at z_0, so given $\varepsilon > 0$, there exists $\delta > 0$:

$$|z - z_0| \le \delta \;\Rightarrow\; |f(z) - f(z_0)| < \varepsilon.$$

Take $r_0 = \delta$. Then $|z - z_0| = \delta$, and

$$|I_2| = \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz \le {}^{\varepsilon}/_{\delta} \,.\, 2\pi\delta = 2\pi\varepsilon .$$

Hence I_2 can be made arbitrarily small by taking r_0 sufficiently small. Since I and I_1 are independent of r_0, I_2 must be too. Therefore $I_2 = 0$, and

$$I = f(z_0)\, I_1 = 2\pi i\, f(z_0).$$

Derivatives of Analytic Functions

Theorem 5.4 Suppose f is analytic inside and on a closed contour C, and z_0 lies inside C. Then

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz .$$

That is, we can take the Cauchy integral formula and formally differentiate with respect to z_0.

Proof We omit this proof. It is not unlike the proof of the Cauchy integral formula.

Note We can similarly prove:

$$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^3}\, dz ,$$

$$\cdots\cdot$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz .$$

A Glimpse and an Example

One of most beautiful parts of course on Complex Functions is the development ofTaylor series. We look at this a little later, but for now, note the appearance here of the terms

$$f(n)(z_0)/_{n!}.$$

Example Evaluate (a) $\int_c \cos z /_z \, dz$, (b) $\int_c \sin z /_{z^2} \, dz$, where C is the unit circle $|z| = 1$ taken in the anti-clockwise direction.

(a) Take $f(z) = \cos z$, $z_0 = 0$ (within C). Since $\cos z$ is entire,

$$I_{(a)} = 2\pi i . \cos z_0 = 2\pi i \text{ (the Cauchy integral formula).}$$

(b) Take $f(z) = \sin z, z_0 = 0, f'(z) = \cos z$. Then

$$I_{(b)} = 2\pi i . f'(z_0) = 2\pi i \cos 0 = 2\pi i \text{ (the Derivative formula).}$$

Another Example

Let us take $f(z) = 1$ in Cauchy's integral formula and the Derivative formulae, and let C be any contour about z_0.

(a) By Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int_c \frac{dz}{z - z_0} = 1.$$

This can be rewritten as

$$\int_c \frac{dz}{z - z_0} = 2\pi i.$$

(b) By the Derivative formulae,

$$\frac{n!}{2\pi i}\int_c \frac{dz}{(z - z_0)^{n+1}} = 0.$$

This can be rewritten as

$$\int_c \frac{dz}{(z - z_0)^{n+1}} = 0, \; n = 1, 2, 3, ...$$

QUIZ 5.4

1. If C is the (positive) contour $|z| = 4$, then

$$\int_C \frac{2z\,dz}{z-2} = $$

2. If C is the (positive) contour $|z| = 1$, then

$$\int_C \frac{2z\,dz}{z-2} = $$

3. If C is the (positive) contour $|z| = 4$, then

$$\int_C \frac{2z\,dz}{(z-2)^2} = $$

4. If C is the (positive) contour $|z| = 1$, then

$$\int_C \frac{2z\,dz}{(z-2)^2} = $$

CHECK

Corollary of the Derivative Formulae

If a function f is analytic at a point, then by the Derivative formulae, its derivatives of all orders are also analytic functions at that point.

Now if $f(z) = u + iv$, then

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

So if $f'(z)$ is analytic, then u_x, v_x, u_y, v_y are all differentiable and so continuous.

In the same way, using $f''(z)$ etc., we see that all partial derivatives of u, v of all orders are continuous at any point where $f(z)$ is analytic. Thus we have:

Corollary If $f = u + iv$ is analytic at a point, then all partial derivatives of u, v of all orders are continuous there.

Note Cauchy's integral formula and the Derivative formulae easily extend to the boundaries of multiply connected domains.

A Useful Lemma

Here is an interesting little result.

Lemma If f is analytic, and $|f|$ is constant, then f is constant.

Proof Let $f = u + iv$. Then we are given that $u^2 + v^2 = c$.
So

$$2uu_x + 2vv_x = 0, \qquad 2uu_y + 2vv_y = 0,$$

leading to

$$u_x / u_y = v_x / v_y .$$

Also, by the Cauchy-Riemann equations, $u_x = v_y, \quad u_y = -v_x$.

So, eliminating the u terms, we get $v_x^{\,2} + v_y^{\,2} = 0$
and so

$$v_x = 0 = v_y = u_x = u_y .$$

Hence $f'(z) = 0$ and so $f(z)$ is constant.

Maximum Modulus

Let f be analytic and not constant on the open disk $|z - z_0| < r_0$, centred at z_0. If C is any circle $|z - z_0| = r$ $(0 < r < r_0)$ then by the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \qquad (*)$$

Path C is in the positive sense and parametrizes: $z(\theta) = z_0 + r\exp(i\theta)$ $(0 \le \theta \le 2\pi)$. So (*) becomes

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta .$$

This means that the value at the centre is the arithmetic mean of the values on the circle. Thus

$$|f(z_0)| = \frac{1}{2\pi i} \left| \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \right| \le \frac{1}{2\pi i} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta$$

$$(0 \le r < r_0). \qquad (+)$$

Now suppose $|f(z_0)|$ is a maximum. Then $|f(z)| \le |f(z_0)|$ for all $z : |z - z_0| < r_0$.

[*Continued*]

Maximum Modulus Principle

So, $\frac{1}{2\pi i}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \leq |f(z_0)| \qquad (0 \leq r < r_0). \quad (+)$

Combining the two inequalities (+), we have

$$|f(z_0)| = \frac{1}{2\pi i}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta$$

From this we can show that, $|f(z)| = |f(z_0)|$ for all $z : |z - z_0| < r_0$.

For, write the above equation as

$$\int_0^{2\pi} |f(z_0 + re^{i\theta}) - f(z_0)|\, d\theta = 0.$$

Since the integrand is non-negative, we deduce it must be 0 for all $z : |z - z_0| < r_0$.
This shows that $f(z) = f(z_0)$ for all z in the disk. So f is constant in the disk.
This is a contradiction.

Theorem 5.5 (Maximum Modulus Principle) If f is analytic and not constant in the interior of a region then $|f(z)|$ has no maximum value in that interior.

Some Observations

We deduce that if a function f is continuous in a closed bounded region R and is analytic and not constant in the interior of R, then $|f(z)|$ assumes its maximum value on the boundary of R and never in its interior.

Now let f be analytic in and on the circle C_0 defined by $|z - z_0| = r_0$, and traversed in a positive sense. Then by the Derivative formulae

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_o} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad n = 0, 1, 2, \ldots .$$

If $|f(z)| \leq M$ on C_0, then using the 'ML' bound,

$$|f^{(n)}(z_0)| \leq n!\, M / r_0^{\,n} \quad (n = 0, 1, 2, \ldots),$$

and for $n = 1$

$$|f'(z_0)| \leq M / r_0.$$

Liouville's Theorem

The preceding observations lead to the following theorem.

Theorem 5.6 (Liouville) If f is entire and bounded for all values of z in the complex plane, then $f(z)$ is constant.

Proof By assumption $|f(z)| \leq M$ for all z.

Therefore, as before,

$$|f'(z_0)| \leq M / r_0$$

for each z_0 in the plane, and for any positive r_0, no matter how large.

It follows that $f'(z_0) = 0$. But z_0 is arbitrary.

Thus for all z, $f'(z) = 0$, and so $f(z) =$ constant.

The Fundamental Theorem of Algebra

Theorem 5.7 (Fundamental Theorem) Any polynomial

$$P(z) = a_0 + a_1 z + \ldots + a_n z_n \quad (a_n \neq 0)$$

where $n \geq 1$, has at least one zero. That is, there is at least one point $z_1 : P(z_1) = 0$.
It is curious that this important result has no easy algebraic solution.

Proof Suppose $P(z) \neq 0$ for any z. Then $f(z) = {}^1/_{P(z)}$ is entire.

In fact $f(z)$ is also bounded. For f is continuous and so bounded in any closed disk centred at the origin. Further, if R is large and z is exterior to the disk $|z| \leq R$, then

$$|f(z)| = \frac{1}{|P(z)|} \approx \frac{1}{R^n} \qquad \text{(or worse!)}$$

so f is bounded for all values of z in the plane. (We can tidy up this last argument, but the idea is that when $|z|$ is large, so is $|P(z)|$.)

Now by Liouville's Theorem, $f(z)$ and so $P(z)$ is constant.
This contradiction shows that $P(z)$ has at least one zero.

Factorization of Complex Polynomials

Let z_1 be the zero guaranteed by the Fundamental Theorem.

Then

$$P(z) = (z - z_1)Q(z),$$

where $Q(z)$ is a polynomial of degree $n - 1$.

We deduce (by induction) that

$$P(z) = c(z - z_1)(z - z_2) \ldots (z - z_n).$$

Corollary Any polynomial of degree n, where $n \geq 1$, can be expressed as a product of n linear factors. That is,

$$P(z) = c(z - z_1)(z - z_2) \ldots (z - z_n).$$

where c and the z_k are complex constants.

You might like to compare this result with what happens for polynomials over the reals R.

QUIZ 5.5A

1. If $f = u + iv$ is entire, then all partial derivatives of u, v are continuous everywhere.

(a) True ✔ ; (b) False ☐ .

2. If f is analytic on $|z| \leq 1$, and f has maximum 6 on this disk, then

(a) $f(0) < 6$ ✔ ; (b) $f(0) > 6$ ☐ .

3. We can write any real, degree n polynomial as the product of n linear real factors.

(a) True ✔ ; (b) False ☐ .

4. We can write any real, degree n polynomial as the product of n linear complex factors.

(a) True ✔ ; (b) False ☐ .

CHECK

QUIZ 5.5B

Lemma If f is analytic, and $|f|$ is constant, then f is constant.

Proof Let $f = u + iv$. Then we are given that **{ 1 }**. So **{ 2 }**, $2uu_y + 2vv_y = 0$, leading to **{ 3 }**.

Also, by the Cauchy-Riemann equations, $u_x = v_y$, $u_y = -v_x$.
So, eliminating the u terms, we get $v_x^2 + v_y^2 = 0$ and so $v_x = 0 = v_y = u_x = u_y$.
Hence **{ 4 }** and so $f(z)$ is constant.

Match the above boxes 1, 2, 3, 4 with the selections

(a) $f'(z) = 0$; (b) $u_x / u_y = v_x / v_y$;
(c) $2uu_x + 2vv_x = 0$; (d) $u^2 + v^2 = c$.

My solutions:

1. ☐ **2.** ☐ **3.** ☐ **4.** ☐

CHECK