6. SERIES

Convergence of series

A sequence $z_1, z_2, \ldots, z_n, \ldots$ has **limit** z if for all $\varepsilon > 0$ there exists $N : n > N \Rightarrow | z_n - z | < \varepsilon$.

That is, we can make z_n arbitrarily close to z by taking n sufficiently large. This definition is formally the same as for the real case. Of course, here the points of the sequence lie in the complex plane, rather than on the real line.

The limit, if it exists, is unique.

We say the sequence **converges** to z_0 and write $z_n \to z$ or $\lim_{n\to\infty} z_n = z$.

If there is no limit, we say that the sequence **diverges**.

A First Convergence Theorem

Theorem 6.1 If $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Proof

(⇒) Given $\varepsilon > 0$ there exists $N : n > N \Rightarrow |x_n + iy_n - (x + iy)| < \varepsilon$.

Hence $n > N \Rightarrow |x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$.

i.e. $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

(⇐) Given $\varepsilon > 0$ there exist N_1, N_2 :

$n > N_1 \Rightarrow |x_n - x| < \varepsilon/_2$, $n > N_2 \Rightarrow |y_n - y| < \varepsilon/_2$.

So $n > \max(N_1, N_2) \Rightarrow |x_n - x| + |y_n - y| < \varepsilon$.

Now $|x_n + iy_n - (x + iy)| \leq |x_n - x| + |y_n - y| < \varepsilon$.

So $\lim_{n\to\infty} z_n = z$.

Infinite Series and Partial Sums

The expression

$$z_1 + z_2 + z_3 + \dots$$

is an **infinite series**.

The set

$$S_N = z_1 + z_2 + \dots + z_N$$

is a **partial sum** of the series.

If the sequence $S_1, S_2, \dots, S_N, \dots$ converges to limit S we write $\lim_{n\to\infty} z_n = S$, and S is the **sum** of the series. In this cae we say that the infinite series **converges**.

The sum, when it exists, is unique.

When a series does not converge, it **diverges**.

Convergence of Complex Series

Theorem 6.2 Suppose that $z_n = x_n + iy_n \ (n = 1, 2, \ldots)$ and $S = X + iY$.

Then

$$\sum_1^\infty z_n = S \iff \sum_1^\infty x_n = X \text{ and } \sum_1^\infty y_n = Y.$$

Proof Let $S_N = X_N + iY_N$ denote the Nth partial sum,

where $X_N = \sum_1^N x_n$ and $Y_N = \sum_1^N y_n$.

Now

$$\sum_1^\infty z_n = S \iff \lim_{N\to\infty} S_N = S \iff \lim_{N\to\infty} X_N = X \text{ and } \lim_{N\to\infty} Y_N = Y$$

by Theorem 6.1.

Since X_N, Y_N are the partial sums of $\sum_1^\infty x_n$ and $\sum_1^\infty y_n$, the result follows.

Remainder and Power Series

In establishing that a given series has sum S, we define the remainder after N terms to be:

$$R_N = S - S_N.$$

Since $|S - S_N| = |R_N - 0|$, we have $S_N \to S$ iff $R_N \to 0$ as $N \to \infty$.

Hence, a series converges to sum $S \Leftrightarrow$ the sequence of remainders converges to 0.

We shall be particularly concerned with power series.
A **power series** is a series of the form

$$a_0 + \sum_1^\infty a_n(z - z_0)^n = \sum_0^\infty a_n(z - z_0)^n$$

where z_0 and the a_n are complex constants, and z is any number (variable) in a stated region.

We will use the notation $S(z)$, $S_N(z)$, $R_N(z)$ for the sum, partial sum and remainder respectively.

QUIZ 6.1A

1. If $z_n = 2 + {}^i/_n$ then sequence (z_n) converges.

(a) True ✔ ; (b) False ☐ .

2. If $z_n = x_n + iy_n$ and Σz_n converges,
then Σx_n must converge.

(a) True ✔ ; (b) False ☐ .

3. If $z_n = x_n + iy_n$ and Σx_n converges,
then Σz_n must converge.

(a) True ✔ ; (b) False ☐ .

4. If $\Sigma z_n = S$, then $\Sigma \mathrm{Re}(z_n) = \mathrm{Re}\,(S)$.

(a) True ✔ ; (b) False ☐ .

CHECK

QUIZ 6.1B

Theorem 6.1 If $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Proof

$(\Leftarrow)$ Given $\varepsilon > 0$ there exist N_1, N_2 :

{ **1** } $\Rightarrow$ $|x_n - x| < \varepsilon/_2$, $n > N_2 \Rightarrow$ { **2** }.

So { **3** } $\Rightarrow$ $|x_n - x| + |y_n - y| < \varepsilon$.

Now $|x_n + iy_n - (x + iy)| \le$ { **4** } $< \varepsilon$.

So $\lim_{n\to\infty} z_n = z$.

Match the above boxes 1, 2, 3, 4 with the selections

(a) $n > N_1$, (b) $n > \max(N_1, N_2)$,

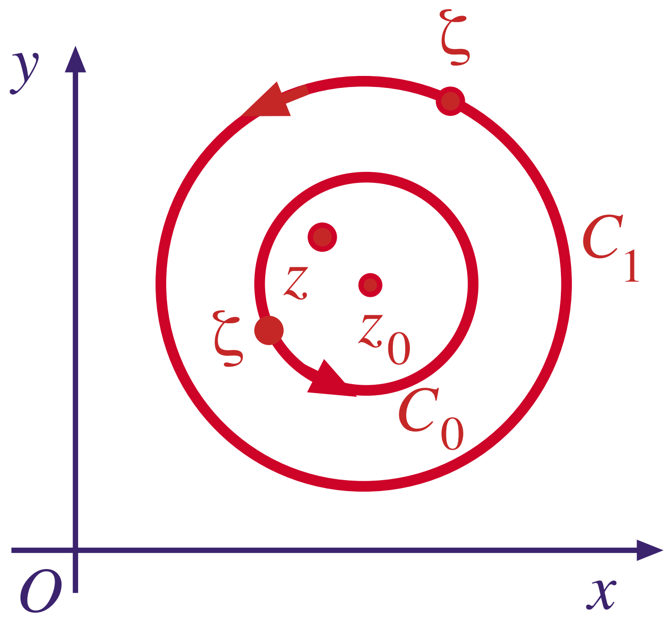
(c) $|y_n - y| < \varepsilon/_2$, (d) $|x_n - x| + |y_n - y|$.

CHECK

My solutions: **1.** ______ **2.** ______ **3.** ______ **4.** ______

Taylor Series

Theorem 6.3 (Taylor) Let f be analytic everywhere inside the circle $C_0 : | z - z_0 | = r_0$. Then at each point z inside C_0



$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)/2!\,.(z - z_0)^2 + \dots + f^{(n)}(z_0)/n!\,.(z - z_0)^n + \dots .$$

That is, the power series converges to $f(z)$ when $| z - z_0 | < r_0$.

Notes

(1) This is the Taylor series expansion about point z_0.

(2) If all terms are real, we get the real Taylor series.

(3) The proof of Taylor's Theorem we give is remarkably 'natural', and is one of the rewards in our study of complex functions.

Proof of Taylor's Theorem (I)

Proof Let z be any fixed point inside C_0 and set $|z - z_0| = r$ (so $r < r_0$). Let C_1 be a circle centred at z_0 and having radius r_1 : $0 < r < r_1 < r_2$. Let ζ (zeta) be any point on this circle; i.e. $|\zeta - z_0| = r_1$.

Now z lies inside C_1, and f is analytic in and on C_1, so by the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

where C_1 is taken in the positive sense. Now

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{(\zeta - z_0)} \cdot \frac{1}{1 - \dfrac{(z - z_0)}{(\zeta - z_0)}}$$

Also for any complex $c \neq 1$,

$$\frac{1}{1 - c} = 1 + c + c^2 + \ldots + c^{N-1} + \frac{c^N}{1 - c}$$

[Continued]

Proof of Taylor's Theorem (II)

So

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\left[1 + \frac{z - z_0}{\zeta - z_0} + \ldots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{N-1} + \frac{\left(\frac{z - z_0}{\zeta - z_0}\right)^{N}}{1 - \left(\frac{z - z_0}{\zeta - z}\right)}\right]$$

Hence

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0} + \frac{f(\zeta)}{(\zeta - z_0)^2}(z - z_0) + \ldots + \frac{f(\zeta)}{(\zeta - z_0)^N}(z - z_0)^{N-1} + \frac{f(\zeta)\,(z - z_0)^N}{(\zeta - z)\,(\zeta - z_0)^N}$$

We next integrate each term anticlockwise around C_1, divide by $2\pi i$, and substitute the Cauchy integral formula and the Derivative formulae:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz,\; n = 0, 1, 2, \ldots .$$

[Continued]

Proof of Taylor's Theorem (III)

So $$f(z) = f(z_0) + f'(z_0)(z - z_0) + \ldots + \frac{f^{(N-1)}(z_0)}{(N-1)!}(z - z_0)^{N-1} + R_N(z)$$

where $$R_N(z) = \frac{(z - z_0)^N}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{(\zeta - z)(\zeta - z_0)^N} \qquad (*)$$

Recall that $|z - z_0| = \mathrm{r},\ |\zeta - z_0| = r_1\ (> r),$

and $|\zeta - z| \geq |\zeta - z_0| - |z - z_0| = r_1 - r.$

Thus if M denotes the maximum value of $|f(\zeta)|$ on C_1, $(*) \Rightarrow$

$$|R_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M.\ 2\pi r_1}{(r_1 - r)\, r_1^{\,N}} = \frac{M r_1}{(r_1 - r)}\left(\frac{r}{r_1}\right)^N.$$

Since $r/r_1 < 1$, $\lim_{n \to \infty} R_N(z) = 0.$

So for each z interior to C_0, the Taylor series for f converges to $f(z)$.

THE END!

Special Case and Observations

Let us seek the Maclaurin expansion for $f(z) = e^z$.

We have $f^{(n)}(z) = e^z$, so $f^{(n)}(0) = 1$.

Also, e^z is analytic for all z, so

$$e^z = 1 + z + z^2/2! + \dots + z^n/n! + \dots = \sum_0^\infty z^n/n!, \quad |z| < \infty.$$

Similarly

$$\sin z = z - z^3/3! + z^5/5! - \dots, \qquad |z| < \infty.$$

$$\cos z = 1 - z^2/2! + z^4/4! - \dots, \qquad |z| < \infty.$$

$$\sinh z = z + z^3/3! + z^5/5! + \dots, \qquad |z| < \infty.$$

$$\cosh z = 1 + z^2/2! + z^4/4! + \dots, \qquad |z| < \infty.$$

Geometric Series

EXPLORATION ...

Let us try to find the Maclaurin series for the function

$$f(z) = \frac{1+2z}{z^2+z^3}$$

Now

$$f(z) = \frac{1+2z}{z^2+z^3} = \frac{1}{z^2}\left(2 - \frac{1}{1+z}\right) = \frac{1}{z^2}(1 + z - z^2 + z^3 - \dots)$$

$$= \frac{1}{z^2} + \frac{1}{z} - 1 + z - \dots$$

This is not a Maclaurin series: the first two terms are unexpected, and the function f has a singularity at $z = 0$.

Question Perhaps there are other interesting series to investigate?

QUIZ 6.2

1. In the Taylor series for f, $z - z_0$ has coefficient

(a) $f(z_0)$ ✔ ; (b) $f'(z_0)$ ☐ ;
(c) $f''(z_0)/2!$ ☐ .

2. A Maclaurin series is a special type of Taylor series.

(a) True ✔ ; (b) False ☐ .

3. $\sin z = 1 - z^2/2! + z^4/4! - \dots$.

(a) True ✔ ; (b) False ☐ .

4. $1/(1 - w^2) = 1 + w^2 + w^4 + \dots$ if $|w| < 1$.

(a) True ✔ ; (b) False ☐ .

CHECK

Laurent's Theorem

Let C_1, C_2 be concentric circles, centre z_0, with radii r_1, r_2 $(r_1 > r_2)$.

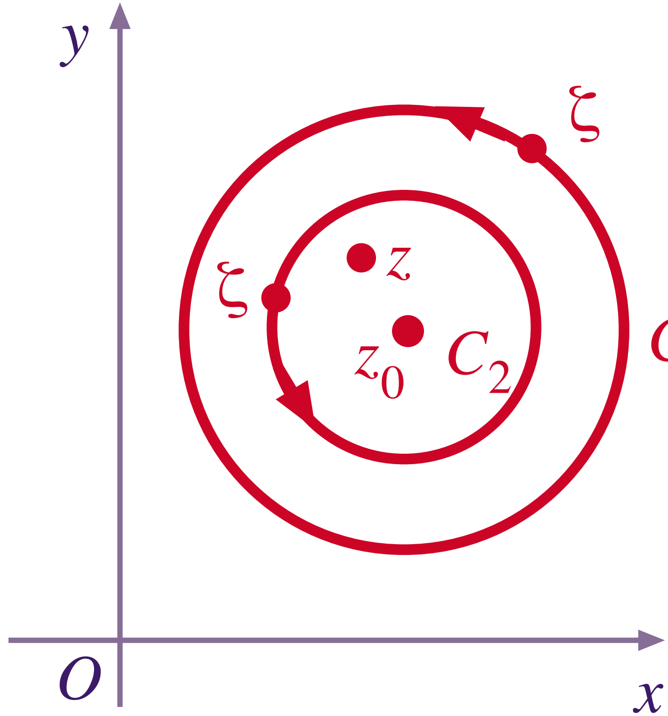
Theorem 6.4 (Laurent) If f is analytic on C_1 and C_2 and throughout the annulus between these two circles, then at each point in this domain, $f(z)$ is represented by the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (*)$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \quad (n = 0, 1, 2, \dots),$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{-n+1}} \quad (n = 1, 2, \dots), \qquad (**)$$



each path of integration taken counter-clockwise.

The series (*) is a **Laurent series**.

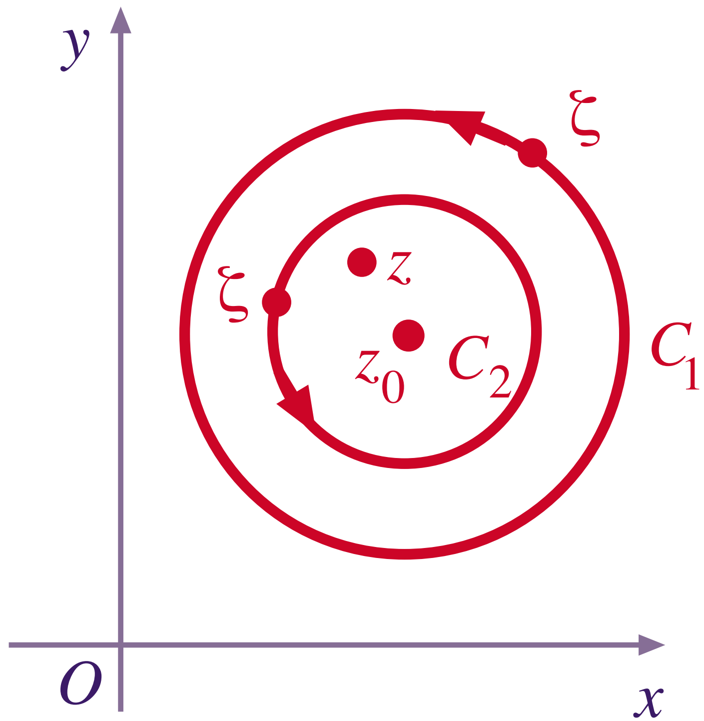
Notes on Laurent's Theorem (I)

(1) If f is analytic at all points in and on C_1 except at z_0, we can take r_2 (radius of C_2) to be arbitrarily small.

Then (*) above is valid for $0 < | z - z_0 | < r_1$.

(2) If f is analytic at all points in and on C_1, then $f(z)/(z - z_0)^{-n+1}$ is analytic in and on C_2 (since $-n + 1 \geq 0$).

So integral (**) is zero, and (*) reduces to the Taylor series.



Notes on Laurent's Theorem (II)

(3) Since $f(z)/(z - z_0)^{n+1}$ and $f(z)/(z - z_0)^{-n+1}$ are analytic throughout the annular region $r_2 \le | z - z_0 | \le r_1$, we can replace $\int_{C_1}$ and $\int_{C_2}$ by $\int_C$ where C is any closed contour around the annulus in the positive direction.

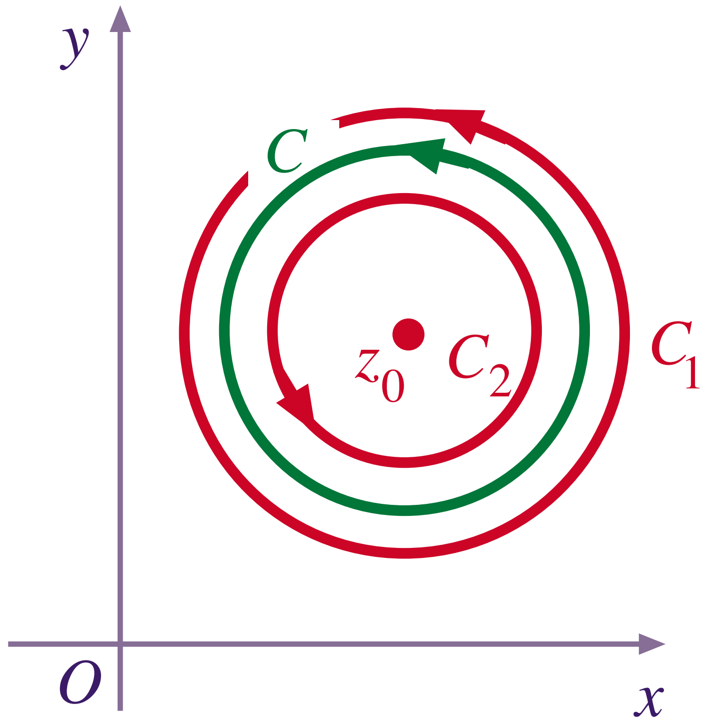
This means that (*) can be written

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

$$r_2 < | z - z_0 | < r_1$$

where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \quad (n = 1, 2, \dots) \qquad (***)$$



Notes on Laurent's Theorem (III)

(4) In practice, some, or even many of the coefficients may be zero.

Example Consider $f(z) = {}^{1}/_{(z - 1)^2}$ where $|z - 1| > 0$.

Here $z_0 = 1$, $c_{-2} = 1$, and all the other coefficients are zero.

Using (***), we observe that

$$c_{-2} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\ d\zeta}{(\zeta - z_0)^{-1}} = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta - z_0} = 1.$$

Notes on Laurent's Theorem (IV)

(5) For particular examples we usually do not find the coefficients of an expansion using the formula. In other words, the general formula is useful more as an existence formula.

Example Find the Laurent series (with $z_0 = 0$) for $f(z) = e^z/z^2$.

Using the Maclaurin expansion for e^z, we obtain:

$$f(z) = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots . \qquad (z \neq 0)$$

Example Find the Laurent series (with $z_0 = 0$) for $f(z) = e^{(1/z)}$.

Using the Maclaurin expansion for e^z with a change of variable, we obtain:

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n} + \ldots . \qquad (z \neq 0)$$

Proof of Laurent's Theorem (I)

If z lies in the annular region, then

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)\, d\zeta}{\zeta - z} \qquad (\dagger)$$

This is the Cauchy integral formula extended to a multiply-connected domain.

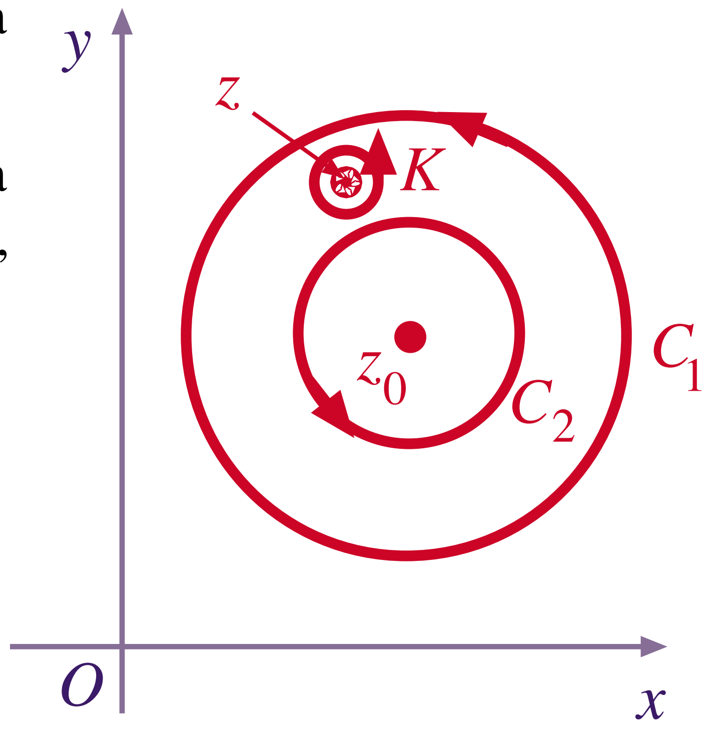
To show this explicitly in this special case, we take a small anti-clockwise directed circle K with centre z, lying within the domain. Then

$$\int_{C_1} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \int_{C_2} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \int_{K} \frac{f(\zeta)\, d\zeta}{\zeta - z} = 0.$$

Notice that by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{K} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

Comparing this equation with equation (†), the validity of (†) is confirmed.



[Continued]

Proof of Laurent's Theorem (II)

The first integral of (†) will give the Taylor series part, so as before

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0} + \frac{f(\zeta)}{(\zeta - z_0)^2}(z - z_0) + \ldots + \frac{f(\zeta)}{(\zeta - z_0)^N}(z - z_0)^{N-1} + \frac{f(\zeta)(z - z_0)^N}{(\zeta - z)(\zeta - z_0)^N}$$

For the second integral of (†) we note that

$$\frac{-1}{\zeta - z} = \frac{1}{(z - z_0) - (\zeta - z_0)} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \dfrac{(\zeta - z_0)}{(z - z_0)}}$$

Multiplying through by $f(\zeta)$, and expanding the last quotient as a geometric series gives

$$\frac{-f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{z - z_0} + \frac{f(\zeta)}{(\zeta - z_0)^{-1}} \cdot \frac{1}{(z - z_0)^2} + \ldots + \frac{f(\zeta)}{(\zeta - z_0)^{-N+1}} \cdot \frac{1}{(z - z_0)^N} + \frac{(\zeta - z_0)^N}{(z - z_0)^N} \cdot \frac{f(\zeta)}{(z - \zeta)}$$

[Continued]

Proof of Laurent's Theorem (III)

So from (†),

$$f(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n + R_N(z) + \sum_{n=1}^{N} \frac{b_n}{(z - z_0)^n} + Q_N(z)$$

where a_n, b_n are as given in the statement of the theorem, $R_N(z)$ is as before, and $R_N(z) \to 0$ as $N \to \infty$.

Also,

$$Q_N(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z - z_0)^N} \int_{C_2} \frac{(\zeta - z_0)^N f(\zeta)}{z - \zeta} \, d\zeta$$

If $r = |z - z_0|$, and r_2 is the radius of C_2, then $r_2 < r$.
Let M be the maximum of $|f(\zeta)|$ on C_2. Then

$$|Q_N(z)| \leq \frac{1}{2\pi r^N} \cdot \frac{r_2^N M . 2\pi r_2}{r - r_2} = \frac{Mr_2}{r - r_2} \cdot \left(\frac{r_2}{r}\right)^N \to 0 \text{ as } N \to \infty .$$

This completes the proof of the theorem.

THE END!

QUIZ 6.3

1. A Laurent series is a special type of Taylor series.
(a) True [✔] ; (b) False [ ] .

2. The general coefficient in the Laurent expansion of f about z_0 is
(a) True [✔] ; (b) False [ ] .

3. The Laurent series for $f(z) = ze^{(1/z)}$ about $z = 0$ is $z + 1 + z/2! + z^2/3! + \ldots$.
(a) True [✔] ; (b) False [ ]

4. The Laurent series for $f(z) = \sinh(-1/z)$ about $z = 0$ is $1/z - 1/3!.1/z^3 + 1/5!.1/z^5 - \ldots$.
(a) True [✔] ; (b) False [ ] .

CHECK

Further Properties of Series

There are many parallels with real series. This follows from the fact that Σz_n is convergent if and only if Σx_n and Σy_n are convergent.

Now if Σx_n and Σy_n are convergent, then $x_n \to 0$, $y_n \to 0$.
We deduce that Σz_n convergent $\Rightarrow$ $z_n \to 0$. Of course, the converse is false!!

So the terms of a convergent complex sequence are bounded:
that is, there exists M: $|z_n| < M$ for all n.

We say that Σz_n is **absolutely convergent** (**AC**) if the series $\Sigma |z_n| = \Sigma \sqrt{(x_n^2 + y_n^2)}$ is convergent.

In this case, by the Comparison Test for real series, $\Sigma |x_n|$ and $\Sigma |y_n|$ are convergent.
Thus $\Sigma x_n, \Sigma y_n$ are AC and so convergent. It follows that Σz_n is convergent.

Thus Σz_n absolutely convergent $\Rightarrow$ Σz_n is convergent.

Absolute Convergence of Power Series

We now prove an important result for power series. Analogues of the next results hold for $\Sigma\, a_n(z - z_0)^n$, but we give this proof for $z_0 = 0$.

Theorem 6.5 If a power series $\Sigma\, a_n z^n$ converges when $z = z_1\ (\neq 0)$, then it is A.C. for all $z : |z| < |z_1|$.

Proof Since $\Sigma\, a_n z_1^{\,n}$ is convergent, for some M we have $|\, a_n z_1^{\,n} \,| < M$ for all n.
We write $|z|/|z_1| = k\ \ (< 1)$.
Then

$$|\, a_n z^n \,| = |\, a_n z_1^{\,n} \,|.|\, z\ /\ z_1 \,|^n < Mk^n.$$

Now the series with terms Mk_n $(k < 1)$ is a real, convergent, geometric series.
So by the Comparison Test,

$$\Sigma\, a_n z^n \text{ is convergent.}$$

Circle of Convergence

Our previous result shows that the set of all points inside some circle centred at the origin is a region of convergence for $\Sigma a_n z^n$.
The largest such circle is the **circle of convergence**.

We note that by the theorem, the series cannot converge at any point z_2 outside this circle. Similarly, if the series $\Sigma b_n / z^n$ converges for $z = z_1$, then it is absolutely comvergent at every point z exterior to the circle centre O passing through z_1. The exterior of some circle centred at O is therefore a region of convergence.

Functions defined by Power Series

The theory starts to get a bit solid (boring!) here, so we settle for some stated results. Nevertheless, these results are important.

Let $S(z) = \Sigma a_n z^n$ over some circle of convergence C_1. Thus S is the function defined by the convergent power series.

Some Stated Results (I)

Then

(1) Function $S(z)$ is continuous at each z interior to C_1.

(2) Function $S(z)$ is analytic at each z interior to C_1.

(3) If C is any contour interior to C_1, then the power series can be integrated term by term, i.e.

$$\int_c S(z)\,dz = \Sigma_0^\infty a_n \int_c z^n\,dz.$$

[If C is a closed contour, then of course we get the value 0 on both sides.]

(4) The power series can be differentiated term by term.

Thus for each z inside C_1,

$$S'(z) = \Sigma_0^\infty n a_n z^{n-1}.$$

Some Stated Results (II)

(5) The Taylor / Laurent series about z_0 for a given function is unique.

Illustration

$$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} \ldots .$$

Even though this series is obtained by substituting z^2 in the series for $\sin z$, it will be the same as the Maclaurin series for $\sin(z^2)$.

(6) Let $f(z) = \Sigma\, a_n z^n, \quad g(z) = \Sigma\, b_n z^n$.

If we formally multiply these series together and collect the coefficients of like powers of z, we get the **Cauchy Product** of the two series:

$$f(z).g(z) = a_0b_0 + (a_0b_1 + a_1b_0)z + (a_0b_2 + a_1b_1 + a_2b_0)z^2 + \ldots + (\Sigma_k\, a_kb_{n-k})\, z^n + \ldots .$$

We now have:

The Cauchy Product of two power series converges to the product of their sums at all points interior to their circles of convergence.

QUIZ 6.4A

1. If $z_n \to 0$, then Σz_n must be convergent.

(a) True ☑ ; (b) False ☐ .

2. If Σz_n is absolutely convergent,
then it must be convergent.

(a) True ☑ ; (b) False ☐ .

3. If $\Sigma\, b_n / z^n$ is convergent for $z = z_1$.
and $|z_2| > |z_1|$, then $\Sigma\, b_n / z_2{}^n$ is convergent.

(a) True ☑ ; (b) False ☐ .

4. The Cauchy product of the series for $\sin z$ and $\cos z$ is:

$$S = z - 4z^3/3! + 16z^5/5! + \dots .$$

(a) True ☑ ; (b) False ☐ .

CHECK

QUIZ 6.4B

Theorem 6.5 If a power series $\Sigma a_n z^n$ converges when $z = z_1 (\neq 0)$, then it is A.C. for all $z : |z| < |z_1|$.

Proof Since $\Sigma a_n z_1^n$ is convergent, for some M we have **{ 1 }** for all n. We write **{ 2 }**. Then

$$|a_n z^n| = \{\ 3\ \} < Mk^n.$$

Now the series with terms **{ 4 }** is a real, convergent, geometric series.

So by the Comparison Test,

$$\Sigma a_n z^n \text{ is convergent.}$$

Match the above boxes 1, 2, 3, 4 with the selections

(a) $Mk^n\ (k < 1)$; **(b)** $|a_n z_1^n|.|z / z_1|^n$;

(c) $|a_n z_1^n| < M$; **(d)** $|z| / |z_1| = k\ (< 1)$.

My solutions:

(1) ____ **(2)** ____ **(3)** ____ **(4)** ____ CHECK

Laurent's Theorem: Example I

For the next examples, set

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}.$$

This function is analytic everywhere except at $z = 1$ and $z = 2$.

Find the Maclaurin series for $f(z)$ valid in $|z| < 1$.

Now

$$f(z) = \frac{1}{(2-z)} - \frac{1}{(1-z)} = \frac{1}{2} \cdot \frac{1}{1 - {}^{z}/_{2}} - \frac{1}{(1-z)}$$

Also $|z| < 1 \Rightarrow |{}^{z}/_{2}| < 1$. Hence

$$f(z) = \frac{1}{2} + \frac{1}{2} \cdot \frac{z}{2} + \frac{1}{2} \cdot \frac{z^2}{4} + \ldots - 1 - z - z^2 - \ldots = \sum_{0}^{\infty} \left[\frac{1}{2} \left(\frac{z}{2} \right)^n - z^n \right]$$

valid for $|z| < 1$.

Thus this is the Maclaurin series for $f(z)$ over the given domain.

Laurent's Theorem: Example II

We are given

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}.$$

Find the Maclaurin series for $f(z)$ valid in $1 < |z| < 2$.

In this case we have $|{}^1/_z| < 1$ and $|{}^z/_2| < 1$.

Now

$$f(z) = \frac{1}{z} \cdot \frac{1}{(1 - {}^1/_z)} + \frac{1}{2} \cdot \frac{1}{(1 - {}^z/_2)}$$

So

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \ldots = \sum_0^\infty \left[\frac{1}{z^{n+1}} + \frac{z^n}{2^{n+1}} \right]$$

valid for $1 < |z| < 2$.

This is the required Laurent expansion.

We observe here that c_{-1} (or b_1) $= 1$.

Evaluation of Integrals

We noted in the previous calculation that c_{-1} (or b_1) $= 1$.

Recall that

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

Thus

$$c_{-1} = \frac{1}{2\pi i}\int_C f(\zeta)\,d\zeta = 1.$$

where C is any simple closed contour around the annulus (taken in the positive direction).

That is,

$$\int_C f(\zeta)\,d\zeta = 2\pi i.$$

This suggests the use of the Laurent series for the evaluation of integrals.

Laurent's Theorem: Example III

Again,
$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}.$$

Find the Maclaurin series for $f(z)$ valid in $|z| > 2$.

Here we have $|{}^2/_z| < 1$ and so $|{}^1/_z| < 1$.

Now
$$f(z) = \frac{1}{z}\cdot\frac{1}{(1-{}^1/_z)} - \frac{1}{z}\cdot\frac{1}{(1-{}^2/_z)}$$

So
$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots - \frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} - \ldots = \sum_0^\infty \left[\frac{1}{z^{n+1}} + \frac{z^n}{2^{n+1}} \right]$$

valid for $|z| > 2$.

Note that the coefficient of z^{-1} is zero. Hence $\int_C f(\zeta)\,d\zeta = 0$ for any small contour C about 0 and exterior to the circle $|z| = 2$.

Note It is possible to develop the whole theory of analytic functions beginning with series, but there is no advantage. The proofs are not easy and motivation is lacking.

Zeros of an Analytic Function (I)

If f is analytic at z_0, there exists a circle centre z_0 within which f is represented by a Taylor series:

$$f(z) = a_0 + \Sigma\, a_n(z - z_0)^n \qquad (\,|\,z - z_0\,| < r_0),$$

where $a_0 = f(z_0)$ and $a_n = f^{(n)}(z_0)/n!$. If z_0 is a zero of f, then $a_0 = 0$.

If in addition

$$f'(z_0) = 0 = f''(z_0) = \ldots = f^{(m-1)}(z_0)$$

but $f^{(m)}(z_0) \neq 0$, then z_0 is a **zero of order *m***.

In this case,

$$f(z) = (z - z_0)^m\, \Sigma\, a_{m+n}(z - z_0)^n \qquad (a_m \neq 0,\ \ |\,z - z_0\,| < r_0)$$

$$= (z - z_0)^m\, g(z) \qquad \text{say,}$$

where $g(z_0) = a_m \neq 0$.

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Zeros of an Analytic Function (II)

Thus $f(z) = (z - z_0)^m g(z)$ where $g(z_0) = a_m \neq 0$.
Since $g(z)$ is represented by a convergent power series, g is continuous at z_0. That is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |g(z) - g(z_0)| < \varepsilon$.
Now choose $\varepsilon = |a_m|/2$.
Then there exists $\delta : |z - z_0| < \delta \Rightarrow |g(z) - a_m| < |a_m|/2$.
It follows that $g(z) \neq 0$ at any point in neighbourhood $|z - z_0| < \delta$.

We have proved:

Theorem 6.6 Let f be analytic at point z_0 which is a zero of f. Then there exists a neighbourhood of z_0 throughout which f has no other zeros, unless $f \equiv 0$.

That is, the zeros of an analytic function are isolated.

QUIZ 6.5

1. If $f(z) = {}^{1}/_{[z(1-2z)]}$ and $|z| < {}^{1}/_{2}$, then the coefficient of z^2 in the Laurent expansion for f is :

2. If $f(z) = {}^{1}/_{[z(1-2z)]}$ and $|z| > {}^{1}/_{2}$,
then the coefficient of ${}^{1}/{z^2}$ in the Laurent
expansion for f is :

3. Function $f(z) = z^3 \cos z$ has a zero at
$z = 0$ of order 3.
(a) True ✔ ; (b) False .

4. Function $f(z) = z^3 \sin z$ has a zero at $z = 0$ of order 3.
(a) True ✔ ; (b) False .

CHECK

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