7. RESIDUES AND POLES

Introduction

Definition If there exists a neighbourhood of z_0 throughout which f is analytic except at z_0 itself, then z_0 is an **isolated singularity** of f.

Example

$$f(z) = \frac{z + 1}{z^3(z^2 + 1)}$$

has three isolated singularities: $z = 0$, $z = - i$.

Contrast $\operatorname{Log} z$ which has a continuous *ray* of singularities.

Now from the Cauchy integral formula and the Derivative formulae,

$$\int_c \frac{f(z)\,dz}{z - z_0} = 2\pi i\, f(z_0), \int_c \frac{f(z)\,dz}{(z - z_0)^2} = 2\pi i\, f'(z_0), \quad \ldots$$

where the integrals are in an anti-clockwise direction about a simple closed contour containing z_0. We observe that the values of these integrals are of the form $2\pi i\,.K$.

Residues

We now change our notation, replacing $f(z)/(z - z_0)$ by $f(z)$. So denote by $f(z)$ a function which is analytic on and inside C except at an isolated singular point z_0 inside C. Then $\int_c f(z)\, dz = 2\pi i . K$, where K is a constant and the integral is once anti-clockwise round C.

Definition $K = \frac{1}{2\pi i} \int_c f(z)\, dz$ is the **residue** of f at the isolated singular point z_0.

Theorem 7.1 (Residue Theorem) Let C be a closed contour within and on which function f is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ interior to C. If K_i denotes the residue of f at z_i, then

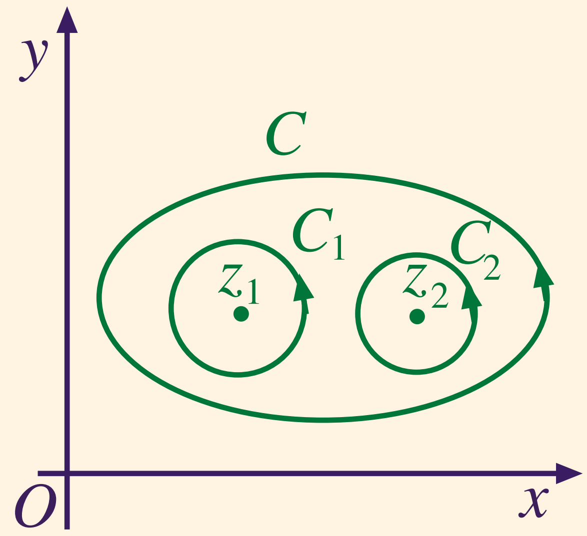
$$\int_c f(z)\, dz = 2\pi i . (K_1 + K_2 + \ldots + K_n),$$

where the integral is around C in positive sense.

Δ

Proof of the Residue Theorem

Proof Let each z_i be enclosed as shown in a circle C_i with radius small enough so that C and the C_i are all separated. Now f is analytic in the remaining region within and including C, so by Cauchy's Theorem:



$$\int_C f(z)\,dz - \int_{C_1} f(z)\,dz - \ldots - \int_{C_n} f(z)\,dz = 0,$$

so

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \ldots + \int_{C_n} f(z)\,dz$$

$$= 2\pi i\,.(K_1 + K_2 + \ldots + K_n),$$

using the definition of K_i.

It follows that evaluation of such integrals depends on our ability to evaluate residues.

Example on the Residue Theorem

Evaluate $\displaystyle\int_C \frac{(5z-2)\,dz}{z(z-1)}$,

where C is $|z| = 2$ taken in the positive sense.

The singularities inside C are $z = 0, 1.$ So integral $I = 2\pi i.(K_0 + K_1)$ (say).

Find K_0. Set $g(z) = {}^{(5z-2)}/_{(z-1)}$ – analytic in a small circle C_0 centred at $0.$
By the Cauchy integral formula, $\int_{C_0} {}^{g(z)}/_{z}\, dz = 2\pi i.g(0) = 2\pi i.2 = 2\pi i.K_0.$

Find K_1. Set $h(z) = {}^{(5z-2)}/_{z}$ – analytic in a small circle C_1 centred at $1.$
Again by the Cauchy integral formula,

$$\int_{C_1} {}^{h(z)}/_{(z-1)}\, dz = 2\pi i.h(1) = 2\pi i.3 = 2\pi i.K_1.$$

Hence $I = 2\pi i.(2+3) = 10\pi i.$

Alternative Solutions

(1) Use partial fractions:

$$\int_C \frac{(5z - 2)\ dz}{z(z-1)} = \int_C \frac{2\ dz}{z} + \int_C \frac{3\ dz}{z-1} = 2\pi i.(2+3) = 10\pi i.$$

(2) Use the Laurent expansion with $|z| > 1$, and find the coefficient of $^1/_z$.

So

$$\frac{(5z-2)}{z(z-1)} = \frac{(5z-2)}{z^2(1 - {}^1/_z)} = \frac{(5z-2)}{z^2}(1 + {}^1/_z + {}^1/_{z^2} + \dots)$$

$$= -\frac{2}{z^2} - \frac{2}{z^3} - \dots + \frac{5}{z} + \frac{5}{z^2} + \dots .$$

Do you always expect the *same* solution? This property is **consistency**.

and noting that the coefficient of $^1/_z$ is 5, we obtain the result as before.

QUIZ 7.1

1. Function $f(z) = \dfrac{z}{(z-1)(z-2)(z^2+4)}$ has just two isolated singularities.

(a) True ☑ ; (b) False ☐ .

2. If f has residues K_1 at $z = 1$ and K_2 at $z = 3i$, C is $|z| = 2$ once positively, then $\int_c f(z)\ dz = 2\pi i.(K_1 + K_2)$.

(a) True ☑ ; (b) False ☐ .

3. With C as in Q2, $\int_c \dfrac{z}{(z-1)(z+i)}\ dz = 2\pi i$.

(a) True ☑ ; (b) False ☐ .

4. If $f(z) = {}^3/_{z^2} - {}^1/_z + 2 + z - 3z^2$ for all $z > 0$, then $\int_c f(z)\ dz = -1$. (C as in Q2.)

(a) True ☑ ; (b) False ☐ .

CHECK

Singularities and Poles

Singularities are obviously important in the theory. The nature of a singularity can be determined from the Laurent expansion

$$f(z) = \sum_{0}^{\infty} a_n (z - z_0)^n + \sum_{1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

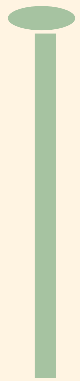
(over some domain), and in particular from the portion involving negative powers, known as the **principal part** of f at z_0.

Suppose $f(z) = \phi(z) / (z - z_0)^n$, where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$.

(This certainly means that ϕ has no $(z - z_0)$ factors).

Then f has a **pole of order** $\boldsymbol{m}$ **at** $\boldsymbol{z_0}$ (or $\boldsymbol{z_0}$ **is a pole**).

A pole of order 1 is a **simple pole**.



Examples of Poles (I)

(1) Consider $f(z) = {(z^2 - 2z + 3)}/{(z - 2)}$.

Now $\phi(z) = z^2 - 2z + 3$ is entire and $\phi(2) \neq 0$, so $z = 2$ is a simple pole.

(2) Consider $f(z) = {(z + 1)}/{(z^2 + 1)^2}$ and $z = -i$.

Set $\phi(z) = {(z + 1)}/{(z - i)^2}$.

Then $f(z) = {\phi(z)}/{(z + i)^2}$, where ϕ is analytic at $z = -i$ and $\phi(-i) \neq 0$.
So $z = -i$ is a pole of order 2.

(3) Consider $f(z) = {\sin z}/{z}$.

Here $\phi(z) = \sin z$ is analytic, but $\phi(0) = 0$. Hence this is not a simple pole. In fact

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^4}{5!} - \ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \ldots$$

We say $z = 0$ is a **removable singularity**. We could define $f(0) = 1$.
(This would give a new function, coinciding with f for $z \neq 0$, but also defined at $z = 0$ and continuous there.)

Examples of Poles (II)

(4) Consider $f(z) = e^{1/z}$. There is a problem here at $z = 0$. Assuming

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \ldots , \qquad (*)$$

it is not possible to put this in the form $\phi(z)/z^m$ for any m.
We call this an **essential singularity**.

(5) Essential singularities often exhibit strange behaviour.
When f has a pole at z_0, we expect $f(z) \to \infty$ as $z \to z_0$. This occurs.
But it may not occur for an essential singularity.

Example Consider the function (*) above.

For this function there is an essential singularity at $z = 0$.

Take $z = r \operatorname{cis} \theta$, so $\exp(1/z) = \exp[(1/r) \operatorname{cis}(-\theta)]$.

Now take $\theta = \pi/2$ and let $r \to 0$ (i.e. approach O along the positive imaginary axis.)

Approaching O along this path, $|e^{1/z}| = \exp[(1/r) \cos \pi/2] = e^0 = 1$.

That is, $e^{1/z}$ does not tend to 0.

Notes on Poles

(1) We can give general formulae for the residues for poles of order m – essentially using Theorems 6.3, 6.4.

(2) Work on series is useful here.

(3) Most examples treat poles of low order.

It is suggested that you learn the Cauchy integral formula and the Rules on Differentiation with respect to z_0. Thus:

$$\frac{1}{0!} f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0}\, dz$$

$$\frac{1}{1!} f'(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^2}\, dz$$

$$\frac{1}{2!} f''(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - z_0)^3}\, dz$$

etc.

More Examples on Poles and Residues

(1) Consider

$$g(z) = \frac{e^{-2z}}{z^3}.$$

This function has a pole of order 3 at $z = 0$.
Hence the residue there is ${}^{1}/_{2!}.f''(0) = {}^{1}/_{2!}.4.e^{0} = 2.$

(2) Consider

$$h(z) = \frac{z+1}{z^2+9}$$

This function has a simple pole at $z = 3i$. The residue there is

$$\left.\frac{z+1}{z+3i}\right|_{z=3i} = \frac{3i+1}{6i} = \frac{3-i}{6}.$$

Exx

QUIZ 7.2

1. If $f(z) = \Sigma_0^\infty a_n z^n + \Sigma_1^\infty b_n z^{-n}$ then $\Sigma_0^\infty a_n z^n$ is the principal part of f at 0.

(a) True ☑ ; (b) False ☐ .

2. If $f(z) = {}^{z}/_{(z-2)}$, then f has a simple pole at $z = 2$.

(a) True ☑ ; (b) False ☐ .

3. If $f(z) = {}^{(\cos z - 1)}/_{z}$, then f has a removable singularity at $z = 0$.

(a) True ☑ ; (b) False ☐ .

4. The residue of $f(z) = \dfrac{z+1}{z^2-9}$ at $z = 3$ is ${}^2/_3$.

(a) True ☑ ; (b) False ☐ .

CHECK

Improper Real Integrals : Cauchy Principal Value

In the case of real improper integrals, we make the definition:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{R \to \infty} \int_{0}^{R} f(x)\ dx + \lim_{R' \to \infty} \int_{-R'}^{0} f(x)\ dx \qquad (1)$$

where both integrals on the right exist. Notice that the variables R, R' tend to infinity independently in the two integrals.

It is useful to define the **Cauchy Principal Value** (**Cauchy P.V**.) in the following way:

$$P.V. \int_{-\infty}^{\infty} f(x)\ dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)\ dx \qquad (2)$$

So with the Cauchy P.V., we are insisting that the upper and lower infinite limits are approached at the same rate.

If the improper integral defined by (1) converges, then the value obtained is the same as the Cauchy P.V.

On the other hand, the Cauchy P.V. (2) may exist and integral (1) not converge.

Using the Cauchy Principal Value

Example Let $f(x) = x$. Here the Cauchy P.V. is 0, but the integral is not convergent.

Special case If f is even and the Cauchy P.V. exists, then $\int_{-\infty}^{\infty} f(x)\, dx$ converges. For in this case

$$\int_0^R f(x)\, dx = \int_{-R}^0 f(x)\, dx = \frac{1}{2}\int_{-R}^R f(x)\, dx$$

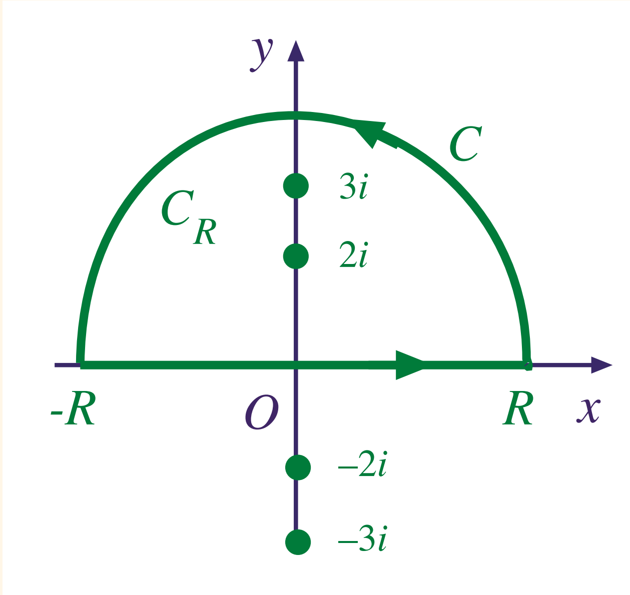
and the existence of the last Cauchy P.V. guarantees the existence of the first two integrals.

Example If $f(x) = {}^{p(x)}/_{q(x)}$ where p, q are real polynomials with no common factors, $q(x)$ has no real zeros, and the degree of $q(x)$ is greater than or equal to the degree of $p(x) + 2$, then $\int_{-\infty}^{\infty} f(x)\, dx$ converges.

Its value can easily be found using residues.

CPV

Cauchy Principal Value : Example (I)



Evaluate $I = \int_0^\infty \frac{x^2 \, dx}{(x^2+9)(x^2+4)^2}$.

Consider $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$.

This has simple poles $z = \pm 3i$; poles of 2nd order $z = \pm 2i$. We find the residues for the poles lying inside the illustrated contour C.

For $z = 3i$: $K_1 = \frac{z^2}{(z+3i)(z^2+4)^2}\Big|_{z=3i} = \frac{3i}{50}$.

For $z = 2i$: $K_2 = \frac{d}{dz}\left(\frac{z^2}{(z^2+9)(z^2+2i)^2}\right)\Big|_{z=2i} = \ldots = \frac{13i}{200}$.

Hence

$$\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i\,(K_1 + K_2) = \frac{\pi}{100} \cdot$$

[Continued]

Cauchy Principal Value : Example (II)

Now on C_R,

$$|f(z)| = \left| \frac{z^2}{(z^2+9)(z^2+4)^2} \right| \le \frac{R^2}{(R^2-9)(R^2-4)^2}$$

and the length of C_R is πR. Thus

$$\left| \int_{C_R} f(z)\, dz \right| \le \frac{\pi R^3}{(R^2-9)(R^2-4)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

So $\int_{-\infty}^{\infty} f(z)\, dz = \pi/100$ (noting that f is even), or $I = \pi/200$.

Question Is this easier than factorizing and using partial fractions in the real case?!

Probably yes, especially as the integral around C_R will fairly clearly always vanish when the difference in degree is 2 or more.

Improrper Integrals Involving Trigonometric Functions

Residue theory is also useful for evaluating integrals of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin x \; dx, \quad \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos x \; dx \qquad (*)$$

where p, q are real polynomials and $q(x)$ has no real zeros.

Note that the previous method cannot be used here. For we have

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad \text{and} \quad |\cos z|^2 = \cos^2 x + \sinh^2 y,$$

so $|\cos z|$ and $|\sin z|$ increase like $\sinh y$ as $y \to \infty$.

∞

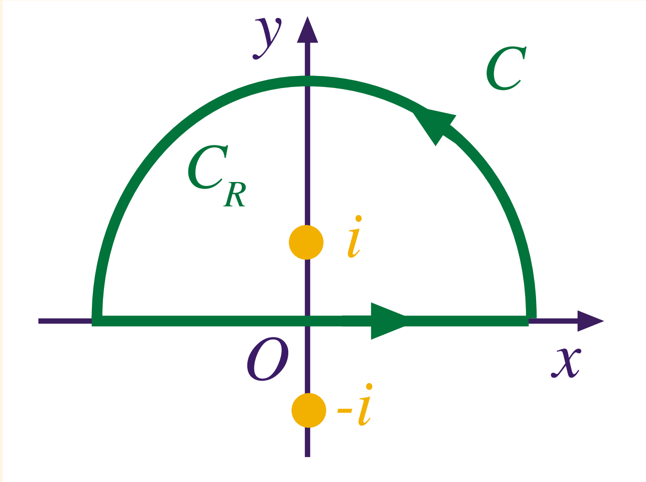
However, the integrals (*) can be combined to give

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} \; dx \; ,$$

and $|e^{iz}| = e^{-y}$, which is bounded in the upper half plane.

Trigonometric Function Integral : Example (I)

Show that
$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx = \frac{\pi}{e}.$$



This integral is the real part of
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx,$$

and we obtain this by integrating $f(z) = e^{iz}/(z^2+1)^2$ along the real axis.

The function f is analytic except for poles of order 2 at $z = -i$. The pole $z = i$ lies inside the illustrated semicircular contour. So
$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\, dx + \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\, dz = 2\pi i . K_1.$$

Now $K_1 = \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\Big|_{z=i} = \frac{i}{2e}$ (calculating).

[Continued]

Trigonometric Function Integral : Example (II)

We show that the second integral tends to 0 as $R \to \infty$. For z in C_R,

$$| z^2 + 1 |^2 \geq (R^2 - 1)^2 \quad \text{and} \quad | e^{iz} | = | e^{-y} | \leq 1 \qquad (y \geq 0)$$

so $$\left| \int_{C_R} \frac{e^{iz}}{(z^2 + 1)^2} dz \right| \leq \frac{1 . \pi r}{(R^2 - 1)^2} \to 0 \text{ as } R \to \infty.$$

Hence

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ix}}{(x^2 + 1)^2} \, dx = \frac{\pi}{e} .$$

So, taking real parts,

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\cos x}{(x^2 + 1)^2} \, dx = \frac{\pi}{e} .$$

Since the integrand is even here, this Cauchy P.V. is the required integral.

Definite Integrals of Trigonometric Functions

We can use residues to evaluate certain definite integrals of the type

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\ d\theta.$$

The variation of θ from 0 to 2π suggests that we use θ as the argument of a point z on the unit circle C. That is, $z = \exp(i\theta) = \sin\theta + i\cos\theta$ $(0 \leq \theta \leq 2\pi)$. Then

$$\sin\theta = \frac{z - \overline{z}}{2i} = \frac{z - {}^1/_z}{2i}; \quad \cos\theta = \frac{z + \overline{z}}{2} = \frac{z + {}^1/_z}{2};$$

and the integral becomes $\int_c F\left(\frac{(z - 1/z)}{2i}, \frac{(z + 1/z)}{2}\right) dz.$

That is, a contour integral of a function of z around the unit circle C taken in the positive sense.

Integrals of Trigonometric Functions : Example (I)

Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{(1 - a^2)}}$ $\quad (|a| < 1,\ a \text{ real})$

The formula is clearly valid for $a = 0$. Suppose that $a \neq 0$. Then

$$I = \int_c \frac{1}{1 + a\left(\dfrac{z - {}^1/_z}{2i}\right)} \frac{dz}{iz} = \int_c \frac{{}^2/_a}{z^2 + ({}^{2i}/_a)z - 1}\, dz$$

where C is the circle $|z| = 1$ traversed in the positive direction.

The denominator has zeros:

$$z_1 = \frac{i}{a}\left(-1 + \sqrt{(1 - a^2)}\right),\quad z_1 = \frac{i}{a}\left(-1 - \sqrt{(1 - a^2)}\right).$$

Hence the integrand is $\displaystyle\frac{{}^2/_a}{(z - z_1)(z - z_2)}$.

[Continued]

Integrals of Trigonometric Functions : Example (II)

Also, noting that $|a| > 1$, we have

$$|z_2| = \frac{(1 + \sqrt{(1 - a^2)})}{|a|} > 1;$$

that is, z_2 lies outside C.

Further, $|z_1 z_2| = 1$, so $|z_1| < 1$, – a simple pole inside C.

The corresponding residue K_1 is:

$$\left.\frac{2/a}{z - z_2}\right|_{z = z_1} = \frac{2/a}{z_1 - z_2} = \frac{1}{i\sqrt{(1 - a^2)}}$$

Hence

$$I = 2\pi\, i . K_1 = \frac{2\pi}{\sqrt{(1 - a^2)}}.$$

THE END

QUIZ 7.3

1. To show that $\int_{-\infty}^{\infty} f(x)\,dx$ converges, it is sufficient to show that P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

(a) True ✔ ; (b) False ☐ .

2. If f is an odd function and P.V.$\int_{-\infty}^{\infty} f(x)\,dx$ exists, then $\int_{-\infty}^{\infty} f(x)\,dx$ must converge.

(a) True ✔ ; (b) False ☐ .

3. $\int_{0}^{\infty} \frac{3x^2+1}{(x^2+3)(x^4-1)}\,dx$ converges.

(a) True ✔ ; (b) False ☐ .

4. To evaluate $\int_{0}^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$ as an integral $\int_C f(z)\,dz$, we use the substitution $z = \exp(i\theta)$.

(a) True ✔ ; (b) False ☐ .

CHECK

Final Comment

Because of time constraints, the course finished here. With a little more time, we would have showed that analytic mappings are **conformal** (preserve angle measure), and worked through a few **boundary value problems**. These would have demonstrated again the practical nature of complex analysis, and given us practice in the use of complex mappings.

I hope you have enjoyed this course, and found the notes helpful. Certain parts of the material may have seemed intricate and fiddly as we worked through them. But if we stand back and look, we find that the complex analysis is very practical, and at the same time beautiful in the way it connects and explains different aspects of our earlier mathematics.

Paul Scott